A.A. VEDENOV

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Introduction

Plasma turbulence, which has been observed by many experimenters in recent years, has been found to be much more complicated than turbulence in ordinary liquids. The reason for this is that collective excitations are readily excited in plasma and have different characteristic frequencies (of the order of the Langmuir, Larmor, etc. frequencies) and different characteristic wavelengths (of the order of the Debye radius, the Larmor radius and so on); in liquids collective oscillations do not take place. The absence of such collective oscillations with the characteristic frequencies and wavelengths in incompressible non-viscous liquids has in fact led to the formulation of the energy distribution in the spectrum of strong turbulence in incompressible liquids [1], which has been confirmed experimentally.

In this monograph we shall review various aspects of the theory of turbulent plasma. In Chapter I we shall summarise the methods available for the description of the dynamics of laminar and weakly turbulent plasma. These methods are used to discuss the properties of collective oscillations (Chapter 2) and to study the stability of the laminar plasma state (Chapter 3). Most of the conclusions

of the linear theory of small oscillations and plasma stability have been confirmed experimentally.

The phenomenon of hysteresis [3], which has been observed in the transition from laminar to turbulent states and vice versa, is discussed in terms of the hard excitation of turbulent pulsations [4] in Chapter 4.

The presence of collective oscillations in plasma enables us to develop a systematic theory of weakly turbulent plasma states in which the wavelengths of turbulent pulsations cover a broad interval of values, but the energy of the pulsations is small in comparison with the thermal energy of the plasma particles.

One of the successes of the theory of weakly turbulent plasma has been the development of the quasi-linear theory of tenuous plasma (Chapter 5) which has predicted in particular the existence of the plateau on the particle distribution functions [5] which has been confirmed experimentally [6].

Further development of the theory requires the introduction of the interaction between plasma oscillations in weakly turbulent plasma. This is considered in Chapter 6 which gives the theory of the self-consistent wave interactions [6], and in Chapter 7 which is devoted to the kinetic equations for the plasma-wave distribution functions.

Chapter 8 puts forward a principle which can be used to determine the spectral energy density in pulsations in strongly turbulent plasma states: the amplitude of the particle velocity in the pulsations must be of the same order of magnitude as the phase velocity of the waves whose growth has led to the appearance of the given turbulent state. We note that the application of this principle to turbulent gravitational waves on the sea surface leads to an energy distribution in pulsations which is in accordance with experiment (Chapter 8).

Chapter 9 is concerned with problems connected with the determination of the transport coefficients in turbulent plasma (effective electrical conductivity and diffusion coefficients).

The last chapter, Chapter 10, is devoted to the scattering of electromagnetic waves in turbulent plasma - an effect which is interesting both for the study of the ionosphere and for laboratory plasma investigations.

Chapter 1

Methods of Description of Plasma

Plasmas consist of electron and ion gases (there may be more than one type of ion) and may often be described by distribution functions $f(\mathbf{x}, \mathbf{v}, t)$ for each type of charge. We shall confine our attention to the 'gas' approximation, i.e. we shall consider the plasma to be an almost ideal gas, which is valid if the Debye radius for the ions (electrons) is considerably greater than the average distance between particles. However, many of the results pertaining, in particular, to the magnetohydrodynamic approximation will apply more generally. These distribution functions may, in principle, be found by solving a system of kinetic (Boltzmann) equations, taking into account only binary collisions and the action of electric and magnetic fields on the plasma particles [1]. These fields are related by the Maxwell equations to the current and charge densities $e\int f\mathbf{v}d\mathbf{v}$ and $e\int fd\mathbf{v}$.

Simplified mathematical models are often employed in the study of the dynamics of laminar and weakly turbulent plasmas. These allow, with certain assumptions, simpler equations to be used instead of the Boltzmann equation.

If we consider characteristic length L, considerably greater than the mean free path l_c for ions (electrons), we may use the gas-dynamical approximation. Formally, the equations of magnetic gas dynamics consist of the equations for the low (up to third) moments of the ion and electron distribution functions and the Maxwell equations for the self-consistent fields in the assumed quasi-neutral plasma [2]. This last assumption, which is valid for linear dimensions considerably larger than the Debye radius, is definitely valid in the gas-dynamical case.

In the study of magnetohydrodynamic stability, we can consider the plasma to be an ideal fluid, i.e. we can delete from the original magnetohydrodynamic equations all terms connected with dissipative effects. This approximation is reasonable if the processes of interest take place in a time considerably smaller than the following characteristic times.

1. The field diffusion time

$$\tau_H = \frac{4\pi\sigma L^2}{c^2}$$

where L is the characteristic length and σ is the conductivity of the plasma;

2. the characteristic time for 'velocity diffusion'

$$\tau_v = \frac{L^2}{\nu}$$

where ν is the kinematic viscosity

3. the characteristic time for temperature diffusion

$$\tau_T = \frac{L^2}{\chi}$$

where χ is the temperature diffusivity of the plasma.

The corresponding dimensionless parameters are:

1. the Reynolds number

$$\mathrm{Re}_m = \frac{\tau_H}{\tau} = \frac{4\pi\sigma L^2}{c^2\tau}$$

where τ is the scale time;

2. the hydrodynamic Reynolds number

$$\mathrm{Re} = \frac{\tau_v}{\tau} = \frac{L^2}{\nu\tau}$$

3. the Pekla number

$$\mathrm{Pe} = \frac{\tau_T}{\tau} = \frac{L^2}{\chi\tau}$$

Thus for the 'ideal' plasma approximation to hold, it is necessary that

$$\mathrm{Re}_m \gg 1, \quad \mathrm{Re} \gg 1, \quad \mathrm{Pe} \gg 1$$

With this assumption, the 'ideal' magnetohydrodynamic equations take the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0, \tag{1.1a}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \frac{1}{4\pi\rho c}[\nabla \times \mathbf{H}, \mathbf{H}] \tag{1.1b}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{H}) \tag{1.1c}$$

$$p = p(\rho) \tag{1.1d}$$

where (1.1a) is the equation of continuity for the density, (1.1b) is the equation of motion describing the variation in the average velocity $\mathbf{v}$ of an element of plasma under the influence of forces connected with the pressure differential p and ponderomotive forces $\frac{1}{c}\,\mathbf{j} \times \mathbf{H} = \frac{1}{4\pi} \times [\nabla \times \mathbf{H}, \mathbf{H}]$, (1.1c) is the equation for the 'trapping' of magnetic lines of force in an ideally conducting plasma and (1.1d) is the equation of state.

It is convenient to write the ponderomotive force in the form:

$$-\frac{1}{4\pi}\mathbf{H} \times \nabla \times \mathbf{H} = -\nabla \frac{\mathbf{H}^2}{8\pi} + \frac{1}{4\pi}(\mathbf{H} \cdot \nabla)\,\mathbf{H}$$

where the first term is the gradient of the 'magnetic pressure' and the second gives the projection

$$\mathbf{n}\,\frac{1}{4\pi}(\mathbf{H} \cdot \nabla)\,\mathbf{H} = \frac{H^2}{4\pi R}$$

where $\mathbf{n}$ is the normal to a line of force and R is its radius of curvature. In other words, the form of the second term resembles the elastic force in a displaced stretched string and is therefore called the 'tension of magnetic lines of force.'

The introduction of dissipative effects makes the system of equations (1.1) more complicated. The right-hand side of (1.1b) acquires the term

$$\eta \Delta \mathbf{v} + \left(\frac{\eta}{3} + \zeta\right) \nabla\nabla \cdot \mathbf{v}$$

which results from the influence of viscosity (η, ζ are coefficients of viscosity), and the right-hand side of (1.1c) is augmented by the term

$$\frac{c^2}{4\pi\sigma}\Delta\mathbf{H}$$

where c is the velocity of light in vacuum. The latter term represents the effect of the electrical conductivity of plasma. The pressure no longer follows the adiabatic law, and (1.1d) must be replaced by the two equations $p = p(\rho, T)$ and

$$\frac{\partial}{\partial t}\left(\rho\left(\frac{v^2}{2}+c_v T\right)+\frac{\mathbf{H}^2}{8\pi}\right) = -\nabla\left(\rho\mathbf{v}\left(\frac{v^2}{2}+c_p T\right)\right.$$
$$\left.+\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})-(\mathbf{v}\hat{\sigma}')-\varkappa\nabla T\right)$$

where

$$\sigma'_{ik} = \eta\left(\nabla_k v_i+\nabla_i v_k-\frac{2}{3}\delta_{ik}\nabla_l v_l\right)-\zeta\delta_{ik}\nabla_l v_l$$

and $\varkappa$ is the thermal conductivity. σ, ζ, $\varkappa$ are scalar quantities only when the mean free path is considerably smaller than the average Larmor radius for ions (electrons). In a strong magnetic field, the Larmor radius becomes smaller, and the equations become even more complicated because of anisotropy in the transport coefficients.

When the characteristic linear dimensions are much smaller than the mean free path, the collisionless Boltzmann equations may be used to describe the plasma [1]. This is valid since each ion and electron then moves on its own trajectory under the influence of magnetic and electric fields which depend on the motion of all the electrons and ions.

$$\frac{\partial f}{\partial t}+(\mathbf{v}\nabla)f+\left(\frac{e\mathbf{E}}{m}+\frac{e}{mc}\mathbf{v}\times\mathbf{H}\right)\frac{\partial f}{\partial\mathbf{v}} = 0$$

$$\nabla\cdot\mathbf{E} = 4\pi e\left(\int f_i\,d\mathbf{v}-\int f_e\,d\mathbf{v}\right),\quad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$$

$$\nabla\times\mathbf{H} = \frac{4\pi}{c}e\left(\int f_i\mathbf{v}\,d\mathbf{v}-\int f_e\mathbf{v}\,d\mathbf{v}\right)+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \qquad (1.2)$$

$$\nabla\cdot\mathbf{H} = 0$$

These equations can only be used in practice to investigate the stability of the simplest cases of homogeneous or almost homogeneous background, i.e. idealised equilibrium conditions for which the unperturbed distribution of plasma

parameters is a slowly-varying function of the coordinates. This means that the wavelengths of the disturbances under consideration are much smaller than the characteristic linear dimensions of the unperturbed distribution.

The system of equations given by (1.2) may be greatly simplified if the characteristic linear dimensions are much larger than the average Larmor radius for ions (electrons) and the times are somewhat longer than the Larmor period. The trajectory of each charge in the plasma can then be regarded as the superposition of a slow drift at right angles to the lines of force, a motion along the lines of force, and a rapid Larmor rotation. This drift approximation yields simple equations of motion for the centre of the Larmor circle after averaging over the rapid rotation:

$$\frac{d\mathbf{x}}{dt} = v_{\parallel}\frac{\mathbf{H}}{H} + c\frac{\mathbf{E}\times\mathbf{H}}{H^2} + \frac{c}{eH^2}\mathbf{F}\times\mathbf{H}$$

Here $v_{\parallel}$ is the velocity along the lines of force and the second term describes the electric drift. The last term describes the drift under the influence of the force

$$\mathbf{F} = \mu\nabla H - M(\boldsymbol{\varepsilon}_0\cdot\nabla)\,\boldsymbol{\varepsilon}_0 v_{\parallel}^2 - \frac{d}{dt}c\frac{\mathbf{E}\times\mathbf{H}}{H^2}M$$

where $-\mu\nabla H$ is the force acting on a particle with a magnetic moment $\mu = \frac{Mv_{\perp}^2}{2H}$ in a non-uniform magnetic field, $-M(\boldsymbol{\varepsilon}_0\cdot\nabla)\,\boldsymbol{\varepsilon}_0 v_{\parallel}^2 - \frac{d}{dt}c\frac{\mathbf{E}\times\mathbf{H}}{H^2}M$ is the inertial force and M is the particle mass. The magnetic moment μ is conserved in the drift approximation. The equation of motion along a line of force is of the form

$$M\frac{dv_{\parallel}}{dt} = -\mu\frac{(\nabla H,\ \mathbf{H})}{H} + e\frac{\mathbf{E}\cdot\mathbf{H}}{H}$$

In the drift approximation we can replace the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ by $f_{\mathrm{dr}}(v_{\parallel}, \mu, \mathbf{x}, t)$ in which the number of independent variables is less by one. The kinetic equation for $f_{\mathrm{dr}}(v_{\parallel}, \mu, \mathbf{x}, t)$ has the form of a continuity equation in the space $v_{\parallel}$, μ, $\mathbf{x}$:

$$\frac{\partial f_{\mathrm{dr}}}{\partial t} + \nabla_{\mathbf{x}}\left(\frac{d\mathbf{x}}{dt}f_{\mathrm{dr}}\right) + \frac{\partial}{\partial v_{\parallel}}\frac{dv_{\parallel}}{dt}f_{\mathrm{dr}} = 0 \qquad (1.3)$$

The drift kinetic equation (1.3), together with the Maxwell equations for the fields, exhibits a distinctive peculiarity;

when there is no dependence on position along the lines of force, the equations for the moments of f_{dr} are of the same form as the magnetohydrodynamic equations to within the adiabatic exponent γ, which is equal to 2 in the drift approximation. Indeed, $p_\perp \sim nmv_\perp^2 \sim n\mu H \sim n^2$, since μ is a constant and $H \sim n$ because of the 'trapping' of lines of force. Thus, even in the absence of binary collisions, the magnetohydrodynamic equations are valid for motion transverse to the lines of force [3].

The equations of 'two-fluid gas dynamics' are often employed in the study of the dynamics of tenuous plasma. In this model the plasma is considered to be composed of a mixture of two ideal charged gases (electrons and ions), each experiencing the self-consistent electric and magnetic fields (satisfying the Maxwell equations) and the pressure gradient. It is assumed that the pressure tensor is isotropic for both gases. The system of equations of two-fluid gas dynamics takes the form

$$\begin{aligned} &m_\alpha \frac{d\mathbf{v}_\alpha}{dt} = e_\alpha \left(\mathbf{E} + \frac{1}{c}\mathbf{v}_\alpha \times \mathbf{H}\right) - \frac{\nabla p_\alpha}{n_\alpha} \\ &\frac{\partial n_\alpha}{\partial t} + \nabla \cdot n_\alpha \mathbf{v}_\alpha = 0 \\ &\nabla \times \mathbf{H} = \frac{4\pi}{c}\sum e_\alpha n_\alpha \mathbf{v}_\alpha + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}; \quad \nabla \cdot \mathbf{H} = 0 \\ &\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \nabla \cdot \mathbf{E} = 4\pi \sum e_\alpha n_\alpha \end{aligned} \tag{1.4}$$

where the subscript $\alpha = e, i$ denotes the type of particle, n, $\mathbf{v}$ the density and velocity, $\mathbf{E}$, $\mathbf{H}$ the fields and $p_\alpha = n_\alpha T_\alpha$.

If the plasma is not fully ionised, three-fluid gas dynamics is sometimes used to compute the interactions of electrons and ions with the neutral plasma component. The corresponding system of equations is derived from (1.4) by the addition of equations of motion and continuity for the neutral component and the inclusion of a frictional term in all equations of motion.

In the case of slow motions in plasma acted upon by a high-frequency electromagnetic field, it is convenient to use the high-frequency potential, describing (over the period) the average effect of this field on the plasma particles. We will derive an expression for the high-frequency potential in the absence of an external magnetic field. The solution of the equation of motion of a particle in the

high-frequency electromagnetic field

$$\ddot{\mathbf{x}} = \frac{e}{m}\mathbf{E}(\mathbf{x}, t) + \frac{e}{mc}\dot{\mathbf{x}} \times \mathbf{H}(\mathbf{x}, t)$$

will be sought in the form

$$\mathbf{x} = \mathbf{y} + \boldsymbol{\xi}$$

where $\boldsymbol{\xi}$ is the displacement in the high-frequency field and $\mathbf{y}$ is the slow displacement of the particle. Assuming that the fields $\mathbf{E}$ and $\mathbf{H}$ vary slowly with position, and expanding $\mathbf{E}$ and $\mathbf{H}$ in powers of ξ, we find

$$\begin{aligned} \ddot{\mathbf{y}} + \ddot{\boldsymbol{\xi}} = \frac{e}{m}\left(\mathbf{E}(\mathbf{y}, t) + \left(\boldsymbol{\xi} \cdot \frac{\partial}{\partial \mathbf{y}}\right)\mathbf{E}(\mathbf{y}, t) + \ldots\right) \\ + \frac{e}{mc}(\dot{\mathbf{y}} \times \mathbf{H}(\mathbf{y}, t) + \dot{\boldsymbol{\xi}} \times \mathbf{H}(\mathbf{y}, t) + \ldots) \end{aligned} \quad (1.5)$$

Taking the rapidly oscillating term in (1.5), we obtain

$$\boldsymbol{\xi} = -\frac{e}{m}\sum_{\omega}\frac{E_{\omega}(\mathbf{y})}{\omega^2}e^{-i\omega t}$$

Inserting this term in (1.5), substituting $\frac{c}{i\omega}\nabla \times E_{\omega}$ for H_{ω} and averaging over time, we find the average force $\mathbf{F}$ on the particle:

$$\mathbf{F} = \frac{e}{m}\left\langle \boldsymbol{\xi} \cdot \frac{\partial}{\partial \mathbf{y}}\mathbf{E}\right\rangle + \frac{e}{mc}\langle \dot{\boldsymbol{\xi}} \times \mathbf{H}\rangle = -\nabla \sum \frac{e^2\mathbf{E}_{\omega}^2}{2m\omega^2} \quad (1.6)$$

The force acting on the electrons in a unit volume of plasma is thus given by

$$\mathbf{f} = -n\nabla \sum_{\omega}\frac{e^2}{2m\omega^2}\mathbf{E}_{\omega}^2$$

In general, the force acting on the electrons in a unit volume of plasma is [4]

$$\mathbf{f} = \frac{1}{16\pi}\sum_{i,k}(\varepsilon_{ik} - \delta_{ik})\nabla E_i^* E_k$$

where E_k is the complex amplitude of the electric field, E_k^* is its complex conjugate and ε_{ik} is the permittivity tensor of the medium.

It is convenient to employ the equations of quasi-linear plasma theory in the study of processes in weakly turbulent plasma in which the energy contained in the turbulent pulsations is much less than that of the random motion of the particles. These equations include the equation for the growth or damping of the pulsations (plasma waves) and a diffusion-type equation for the distribution function for the particles whose velocity approaches the phase velocity of the waves (see Chapter 5).

The interaction between high- and low-frequency waves in weakly turbulent plasma can be described by the self-consistent equations for the distribution function for high-frequency waves in the coordinate and wave-number space, and hydrodynamic-type equations for the density, velocity and pressure of matter (see Chapter 6).

A useful method, which can be used for weakly turbulent plasma, is the method of kinematic equations for the rate of change of the wave distribution function (relating to one or more oscillatory branches) which is due to the processes of decay, merging and dispersion of the waves (see Chapter 7).

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Chapter 2

Plasma Oscillations

Collective plasma oscillations (waves) play an important role in the theory of plasma stability and turbulence. The properties of small-amplitude waves propagating in uniform plasma are considered below within the framework of two-fluid hydrodynamics [1].

For a plane monochromatic wave, all quantities vary as $e^{-i\omega t+ikx}$, so that we can replace $\partial/\partial t$ by $-i\omega$ and ∇ by $i\mathbf{k}$ in the linearised equations (1.4). Eliminating two quantities; $\mathbf{k}\cdot\mathbf{E}$ and $\mathbf{k}\cdot\mathbf{H}$, with the aid of the time-independent Maxwell equations, we obtain for the remaining quantities (ρ_i, ρ_e, v_i, v_e, $\mathbf{k}\times\mathbf{E}$, $\mathbf{k}\times\mathbf{H}$) a system of homogeneous equations of the form $i\omega f = \ldots$, where the right-hand side does not contain the time. The dispersion relation is obtained by equating the determinant of this system to zero. This relation is of the 12th degree in ω, but since we are not considering any dissipative processes, the equations are reversible. This means that ω^2 rather than ω appears directly in the dispersion relation. In general, we thus obtain six branches for the function $\omega(\mathbf{k})$. It is convenient to introduce the bulk velocity $\mathbf{v}$ and the relative velocity $\mathbf{u}=\mathbf{v}_i-\mathbf{v}_e$ in place of the

separate electron and ion velocities. Expressing all the quantities in terms of **u** and v and substituting these values in (1.4) we obtain the following equations:

$$(\omega^2 - \omega_i^2 - c_e^2k^2)\,\mathbf{u}_\parallel + \frac{\omega^2(\omega^2 - \omega_0^2 - c^2k^2)}{\omega^2 - c^2k^2}\mathbf{u}_\perp - j\omega\Omega_e\mathbf{h}\times\mathbf{u}$$
$$+ i\omega\Omega_e\mathbf{h}\times\mathbf{v}_\parallel + \mu\Omega_e^2\mathbf{h}\times(\mathbf{h}\times\mathbf{u})_\perp + c_e^2k^2\mathbf{v}_\parallel = 0 \tag{2.1}$$

$$(\omega^2 - c_0^2k^2)\,\mathbf{v}_\parallel + \mu i\omega\Omega_e(\mathbf{h}\times\mathbf{u})_\parallel + \mu c_e^2k^2\mathbf{u}_\parallel = 0 \tag{2.2}$$

where $\mathbf{h} = \mathbf{H}_0/H_0$, the symbols $\perp$ and $\parallel$ designate the vector components perpendicular and parallel to **k** respectively, and

$$\omega_0^2 = \frac{4\pi e^2 n_e}{m}, \quad \Omega_e = eH_0/m_e c, \quad c_0^2 = \frac{c_e^2\rho_e^0 + c_i^2\rho_i^0}{\rho_i^0}$$
$$\mu = \frac{\rho_e^0}{\rho_i^0}, \quad \rho = nm, \quad c_\alpha^2 = \gamma_\alpha\frac{T_\alpha}{m_\alpha} \tag{2.3}$$

$\gamma_\alpha \sim 1$ is the effective adiabatic exponent. In the coefficients of the equation, we have neglected terms of the order of μ in comparison with unity. The axes of the coordinate system are chosen so that **k** is parallel to the x axis and $h_y = 0$.

To simplify our calculations we will assume that the electron Larmor frequency is smaller than the Langmuir frequency

$$\Omega_e \ll \omega_0 \tag{2.4}$$

This is valid if the magnetic energy does not greatly exceed the energy of the particles. It ensures that three of the six roots of the dispersion relation are of a higher order of magnitude than the others.

HIGH-FREQUENCY OSCILLATIONS

The motion of ions may be disregarded in the case of high-frequency oscillations. We obtain these oscillations from (2.1) by setting $\mu = 0$, $\mathbf{v} = 0$. The dispersion relation takes the form

$$(\omega^2 - \omega_e^2)(\omega^2 - \omega_c^2)^2 - \Omega_e^2\,[h_x^2\omega^{-2}(\omega^2 - \omega_e^2)(\omega^2c^2k^2)^2$$
$$+ h_z^2(\omega^2 - \omega_c^2)(\omega^2 - c^2k^2)] = 0 \tag{2.5}$$

where

$$\omega_e^2 = \omega_0^2 + c_e^2 k^2, \quad \omega_c^2 = \omega_0^2 + c^2 k^2$$

To obtain simple explicit expressions for the three 'large' roots, we consider the limiting cases of short and long wavelengths. The quantity

$$a = (c^2 / \omega_0 \Omega_e)^{1/2}$$

is the relevant characteristic length.

Short waves

When $ka \gg 1$ Equation (2.5) has the roots

$$\omega^2 = \omega_e^2 + \Omega_e^2 h_z^2 (1 - \omega_e^2 / c^2 k^2) \tag{2.6}$$

$$\omega^2 = \omega_c^2 \pm \omega_0 \Omega_e h_x (1 + c^2 k^2 / \omega_0^2)^{1/2} \tag{2.7}$$

The root given by (2.6) corresponds to longitudinal electron plasma waves. Their energy is concentrated mainly in the electric field, in the kinetic energy of the electrons and in their pressure. Electromagnetic waves in plasma (e.g. light) undergo dispersion as described by (2.7). The two roots of (2.7) represent the two possible circular polarisations. The energy of the oscillations is contained in the electromagnetic field and in the kinetic energy of the electrons.

Long waves

When $ka \ll 1$, the roots of (2.5) are

$$\omega^2 = \omega_e^2 + h_z^2 c^2 k^2 \tag{2.8}$$

$$\omega^2 = \omega_c^2 \pm \omega_0 \Omega_e \tag{2.9}$$

The two oscillations in (2.9) have opposite circular polarisations. In all three cases, the energy of the oscillations is contained mostly in the electric field and the electron kinetic energy. Isotropic plasma, with no magnetic field, exhibits one longitudinal oscillation with $\omega^2 = \omega_e^2$ and two

transverse oscillations with $\omega^2 = \omega_c^2$ and different planes of polarisation. The application of a moderate magnetic field $\Omega_e \ll \omega_0$ leads to the following changes in the spectrum of high-frequency oscillations. The transverse waves split into right and left polarisations, with a frequency separation equal to Ω_e at $k = 0$, which decreases with increasing k and is very small when $k > \omega_0/c$. In the case of longitudinal waves, there is a change in the curvature of the dispersion curve for small k.

LOW-FREQUENCY OSCILLATIONS

Low-frequency oscillations can be obtained by assuming that $\omega \ll \omega_0$, ck, an assumption whose validity is borne out by the final results. By projecting (2.1) on the x axis, one can readily verify that u_x is small. Low-frequency oscillations, then, are quasi-neutral and may be considered to sufficient accuracy by neglecting the space charge and the displacement current. Let

$$\varkappa = \omega_c^2 / c^2k^2 = 1 + \omega_0^2 / c^2k^2$$
$$\Omega = \mu\Omega_e^2 / \varkappa$$

The dispersion equation assumes the form

$$(\omega^2 - \Omega_0^2h_x^2)\,[\omega^4 - \omega^2(\Omega_0^2 + c^2k^2) + \Omega_0^2h_x^2c^2k^2] - (\mu\varkappa)^{-1}\Omega_0^2h_x^2\omega^2(\omega^2 - c_0^2k^2) = 0 \tag{2.10}$$

Exact and simple expressions for the roots of (2.10) can be obtained if $h_x = 1$ or $h_x = 0$.

For waves propagating along the magnetic field ($h_x = 1$) we have

$$[(\omega^2 - \Omega_0^2)^2 - (\mu\varkappa)^{-1}\Omega_0^2\omega^2]\,(\omega^2 - c_0^2k^2) = 0$$

i.e.

$$\omega_1 = \pm c_0 k \tag{2.11}$$

$$\omega_{2.3}^2 \pm (\mu\varkappa)^{-1/2}\Omega_0\omega_{2.3} - \Omega_0^2 = 0 \tag{2.12}$$

The oscillations (2.11) are ion sound waves; the linear dispersion law (2.11) is valid only for ion sound waves whose wavelength is greater than the Debye radius ($kD \ll 1$). In

order to find the dispersion law for $kD \lesssim 1$, we must calculate the deviation of the plasma from quasi-neutrality due to the oscillations, using Poisson's law

$$-k^2\varphi = 4\pi e\,(\delta n_i - \delta n_e) \tag{2.13}$$

where φ is the amplitude of the potential in the wave, δn_i is the perturbation of ion density, and $\delta n_e = n_0(e^{e\varphi/T} - 1) \approx n_0 e\varphi/T$ is the perturbation of electron density. From (2.13) and the equations of motion and continuity for ions,

$$-i\omega v_i = -ik\frac{e}{M}\varphi, \quad \frac{\delta n_i}{n_0} = \frac{v_i}{\omega/k}$$

we obtain

$$\omega = \pm\frac{c_0 k}{\sqrt{1 + k^2D^2}}$$

This dispersion relation is confirmed by experiment [2].

The frequencies $\omega_{2,3}$ correspond to two circularly polarised waves. For long wavelengths the phase velocity of these waves is the same and equal to the Alfvén velocity: $\omega_{2,3} = \pm v_A k$. As the wavelength decreases, the frequency of one of the oscillations in (2.12) increases, approaches Ω_{Hi} and reaches this value when $k \approx k_1 = \sqrt{m/M}\omega_0/c)$. The frequency of the other oscillation increases considerably faster when $k > k_1$ and when $k \approx \omega_0/c$, reaches the value Ω_{He}:

$$\omega = \frac{cH}{4\pi ne}\frac{k^2}{1 + \frac{k^2c^2}{\omega_0^2}} \tag{2.14}$$

When $k_1 \ll k \ll \omega_0/c$, this formula gives the dispersion relation for spiral waves (helical). The latter have been investigated in detail in connection with solid-state physics experiments [3]. Experimental studies of low-frequency waves in plasmas completely confirm the theory [4].

For propagation transverse to the magnetic field ($h_x = 0$) we have

$$\omega^2 = c_0^2k^2 + \Omega_0^2, \quad \omega^2 = 0, \quad \omega^2 = 0 \tag{2.15}$$

To find the frequencies for an arbitrary direction of propagation, we consider two limiting cases.

Long waves

Let $\mu\varkappa \gg 1$; in this case $\omega_0/ck \gg 1$. The roots of the dispersion relation (2.10) are approximately given by

$$\omega^2 = \Omega_0^2 h_x^2 \simeq c_A^2 (\mathbf{h}\cdot\mathbf{k})^2, \; c_A^2 = H^2/4\pi\rho_0 \tag{2.16}$$

$$\omega^2 = \frac{1}{2}\,[c_A^2 + c_0^2 \pm (c_A^4 + c_0^4 + 2c_A^2 c_0^2 (1 - 2h_x^2)^{1/2})]\,k^2 \tag{2.17}$$

The first of these roots corresponds to Alfvén waves, the second to magnetosonic waves.

Short waves

Let $\mu\varkappa \ll 1$; in this case we may have $\omega_0/ck \gtrless 1$. The roots of the dispersion relation are approximately

$$\omega^2 = \Omega_i^2 h_x^2, \text{ where } \Omega_i = eH_0/m_i c \tag{2.18}$$

$$\omega^2 = \frac{1}{2}\Big[\Omega_0^2 h_x^2/\mu\varkappa + \Omega_0^2 + c_0^2 k^2 \pm \sqrt{(\Omega_0^2 h_x^2/\mu\varkappa + \Omega_0^2 + c_0^2 k^2)^2 - 4\,\frac{c_0^2 k^2 \Omega_0^2 h_x^2}{\mu\varkappa}}\,\Big] \tag{2.19}$$

The frequency (2.18) corresponds to waves which may be called gyromagnetic ion waves. The energy of these waves is concentrated, for the most part, in the kinetic energy of the ions, which move along circles in a plane perpendicular to the wave vector and not the magnetic field, thus satisfying the quasi-neutrality condition for a plasma. In a weak magnetic field,($\Omega_e < \varkappa c_0 k$), one of the roots of (2.19) corresponds to waves which approach sound waves with $\omega^2 = c_0^2 k^2$. The other root, $\omega^2 = \Omega_e^2 h_x^2/\varkappa^2$, corresponds to oscillations in which the vectors $\mathbf{H}$ and $\mathbf{u}$ are parallel and circularly polarised. When $\omega_0/ck \gg 1$ the energy of these oscillations resides mostly in the magnetic field, and when $\omega_0/ck \lesssim 1$, in the magnetic field and the kinetic energy of the electrons. It is natural to call these oscillations magnetoelectronic. Again, in a strong field ($\Omega_e > \varkappa c_0 k$) when $h_x^2 \gg \mu\varkappa$, one of the roots $\omega^2 \approx c_0^2 k^2$ corresponds to sound waves, and the other, $\omega^2 \approx \Omega_e^2 h_x^2/\varkappa^2$, to magnetoelectronic oscillations. If the direction of propagation is very close to that of the magnetic field ($h_z \ll \mu\varkappa \ll 1$), the first root tends to zero,

the second to $\Omega_0^2 + c_0^2k^2$. Both oscillations then become considerably more complicated, since both ions and electrons participate significantly in them.

In non-uniform plasma the so-called drift waves may propagate. These are waves whose phase velocity is of the same order of magnitude as the drift velocity of plasma in a magnetic field due to a pressure gradient [5]. To find the dispersion relation for drift waves, we consider the special case of plasma in a uniform magnetic field $\mathbf{H} = (0, 0, H)$, with a density gradient $n = n(x)$ perpendicular to the magnetic field. In the coordinate system in which the steady electric field is zero, the ions and electrons drift in the y direction with a velocity $\pm \frac{cT}{eH}\frac{\nabla_x n}{n}$. Choosing an excitation of the form $e^{i\omega t + ik_y y}$ and using

$$-i\omega\delta n + c\frac{E_y}{H}\nabla_x n = 0 \tag{2.20}$$

we obtain

$$\omega = \frac{cT}{eH}\frac{\nabla_x n}{n}k_y \tag{2.21}$$

where $\delta n = n\frac{e\varphi}{T}$ and $E_y = ik_y\varphi$. Drift waves have been observed in alkali metal plasmas [6].

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Chapter 3

Plasma Stability

Turbulence arises if the laminar plasma state proves to be unstable against the growth of any perturbation (of density, velocity, electric or magnetic field, etc.). In this chapter we consider the most important departures from plasma stability (observed experimentally) owing to disturbance of the equilibrium velocity distribution, which is associated with ordered relative motion of the particles in any non-uniform system (gradient of density, velocity, magnetic field, etc.). For a more detailed exposition, see, for example [1-2].

MICROSCOPIC INSTABILITY OF A 'NON-MAXWELLIAN' PLASMA

A departure from the equilibrium (Maxwellian) distribution may lead to the build-up of waves in plasma. The corresponding instability criterion, i.e. the condition for the sign of the imaginary part of the frequency $\omega = \omega_r + i\omega_i$ to alter, may be obtained by considering the energy interchange between any of the plasma waves arising as a result of the

fluctuation, and the plasma particles. For very small ω_i ($\omega_i \ll \omega_r$), a wave with a given value ω and corresponding wave vector $\mathbf{k}$ is nearly periodic. The energy of the plasma ions (electrons) oscillating in the periodic field of the wave does not change on the average. The only exception involves those particles in the velocity distribution which resonate with the wave. In the absence of a magnetic field in an unperturbed plasma, only particles with velocities approaching the phase velocity ω/k will resonate. The resonance condition is $\omega - \mathbf{k} \cdot \mathbf{v} = 0$. In the presence of a constant external magnetic field there will also be an effective interaction between the wave and those particles in whose own coordinate system the frequency of the wave $\omega' = \omega - k_{\parallel} v_{\parallel}$ will approach the cyclotron frequency $\omega_H = eH/mc$ (or one of its harmonics $n\omega_H$) as a result of the Doppler effect:

$$\omega - k_{\parallel} v_{\parallel} - n\omega_H = 0, \quad n = \pm 1, \pm 2, \ldots.$$

Particles whose velocity components $v_{\parallel}$ along the magnetic field satisfying this condition will be continuously accelerated by the field of the wave, just as ions are accelerated in a cyclotron.

We shall now consider the specific conditions for the build-up of various waves in plasma.

1. In the absence of a constant magnetic field, longitudinal Langmuir electron oscillations may propagate in a uniform plasma. The lower limit for the corresponding phase velocity is of the order of the thermal velocity of the electrons (the associated minimum wavelength is of the order of the Debye radius). The phase velocity of these oscillations increases with increasing wavelength. Consider a Langmuir wave of frequency ω (and phase velocity ω/k). In a coordinate system moving relative to the laboratory system with velocity ω/k, the electrostatic potential takes the form of a stationary sinusoid of amplitude φ_0, i.e. a series of 'wells' and 'barriers' for the electrons. Electrons whose velocity differs considerably from ω/k, will move freely in this periodic field and their energy will be conserved, on the average. However, those whose velocity differs from ω/k by less than $\sqrt{2e\varphi_0/m}$ will be reflected from the potential 'barriers'. The latter electrons may be divided into two groups, one whose velocity is greater than ω/k, and the other whose velocity is less. The first group is

reflected by the barriers and transfers energy to the wave, whereas the second group receives energy from the wave. The wave amplitude will increase if there is an overall transfer of energy from electrons to the field. This will take place if there are more electrons in the first group than in the second, i.e. if

$$\left(\frac{\partial f_0}{\partial v}\right)_{v=\omega/k} > 0$$

In order for this condition to be satisfied, the electron velocity distribution function f_0 must have at least one secondary maximum at a velocity greater than the thermal velocity. If $\partial f_0/\partial v < 0$ everywhere, $\omega_i < 0$, i.e. the wave is damped (Landau damping). The magnitude of ω_i may be obtained (to within a numerical factor) near the stability limit ($\omega_i \ll \omega_r$) by considering energy exchange between the wave and the resonating particles. The rate of change of the wave amplitude is

$$\omega_i \equiv \gamma = \frac{1}{2\mathscr{E}}\frac{d\mathscr{E}}{dt}$$

where $\mathscr{E}$ is the energy density in the wave, equal in our case to the sum of the energy of the electric field and the kinetic energy, $E_0^2/8\pi + \frac{m}{2}\Sigma v_i^2$, where E_0 is the electric field amplitude and v_i is the amplitude of the velocity of the i-th electron in the wave.

For Langmuir oscillations of frequency ω close to ω_0, the kinetic energy is nearly equal to the energy of the electric field, so that

$$\mathscr{E} = \frac{E_0^2}{4\pi} = \frac{k^2\varphi_0^2}{4\pi}$$

The rate of change of the energy density, $d\mathscr{E}/dt$, is the sum of the energy given per unit time to the wave by the electrons in the first group

$$\dot{\mathscr{E}}_1 = n\frac{m}{2}\int\limits_{\omega/k}^{\omega/k+\sqrt{2e\varphi_0/m}} \left[\frac{v^2}{2} - \frac{(2\omega/k - v)^2}{2}\right]\frac{-\omega/k + v}{\lambda} f_0(v)\,dv,$$

and the energy taken by the second group from the wave

$$\dot{\mathscr{E}}_2 = n\frac{m}{2}\int\limits_{\frac{\omega}{k}-\sqrt{2e\varphi_0/m}}^{\omega/k}\left[\frac{v^2}{2} - \frac{(2\omega/k - v)^2}{2}\right]\frac{v-\frac{\omega}{k}}{\lambda} f_0(v)\,dv$$

For small amplitude waves, $e\varphi_0 \ll m(\omega/k)^2$, the integral may be evaluated by expanding $f_0(v)$ near $v = \omega/k$. We then have

$$\frac{d\mathscr{E}}{dt} = \dot{\mathscr{E}}_1 + \dot{\mathscr{E}}_2 \simeq \varphi_0^2\omega_0^3\left(\frac{df_0}{dv}\right)_{v=\omega/k}$$

$$\gamma \sim \frac{\omega_0^3}{k^2}\left(\frac{df_0}{dv}\right)_{v=\omega/k}$$

2. Consider the build-up of oscillations owing to cyclotron resonance. In this case the instability arises because of a group of particles in cyclotron resonance with the wave ($\omega - k_\parallel v_\parallel = n\omega_H$). Consider the simplest type of wave, propagating along a constant magnetic field ($k_\perp = 0$) with transverse polarisation. Particles which will effectively interact with the wave will be those whose velocity is given by

$$v_\parallel = \frac{\omega - \omega_H}{k}$$

To derive the criterion for plasma stability with respect to the growth of transverse waves with $k_\perp = 0$, let us evaluate the work done by the electric field of the wave on the plasma particles:

$$\frac{d\mathscr{E}}{dt} \sim \mathbf{v}\cdot\mathbf{E} \sim \langle v\rangle E$$

where

$$\langle v\rangle = \int f_1 v\,dv$$

and f_1 is the correction to the unperturbed distribution function f_0. This correction is due to the action of the wave on the particles and varies linearly with the wavefield. It is proportional to

$$\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right)\frac{\partial f_0}{\partial \mathbf{v}} \tag{3.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields in the wave and are related by

$$\mathbf{H} = \frac{c}{\omega}\mathbf{k}\times\mathbf{E}$$

Since $\mathbf{E}$ in the transverse wave under consideration is perpendicular to the constant magnetic field $\mathbf{H}_0$, expression (3.1) may be written in the form

$$-Ev_\perp\left[\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial f_0}{\partial \varepsilon_\perp}+\frac{k}{\omega}\frac{\partial f_0}{\partial (mv_\parallel)}\right] \tag{3.2}$$

(assuming that $f_0 = f_0(\varepsilon_\perp, v_\parallel)$, where $\varepsilon_\perp = mv_\perp^2/2$).

Since it is electrons with longitudinal velocities satisfying the condition $v_\parallel = (\omega - \omega_H)/k$ which interact effectively with the field, we must substitute $v_\parallel = (\omega - \omega_H)/k$ in (3.2). Integrating (3.2) with respect to $\varepsilon_\perp$, we obtain the condition for energy to be given up to the wave by the electrons. Instability occurs when

$$\int \varepsilon_\perp\left(\left(1-\frac{k_\parallel v_\parallel}{\omega}\right)\frac{\partial f_0}{\partial \varepsilon_\perp}+\frac{k}{\omega}\frac{\partial f_0}{\partial (mv_\parallel)}\right)_{v_\parallel=\frac{\omega-\omega_H}{k}} d\varepsilon_\perp > 0 \tag{3.3}$$

Consider, for example, a 'non-isotropic Maxwellian distribution' with different temperatures $T_\parallel$ and $T_\perp$:

$$f_0 \sim e^{-\frac{\varepsilon_\perp}{T_\perp}-\frac{mv_\parallel^2}{2T_\perp}}$$

The instability criterion (3.3) will now take the form

$$\frac{T_\perp}{T_\parallel}+\frac{\omega_H}{\omega}\left(1-\frac{T_\perp}{T_\parallel}\right)<0 \tag{3.4}$$

If the plasma is isotropic $T_\perp = T_\parallel$, and the second term vanishes; if it is only slightly non-isotropic, i.e $\left|1-\frac{T_\perp}{T_\parallel}\right| \ll 1$, instability may arise only for waves with frequency ω much lower than the cyclotron frequency ω_H.

Transverse waves propagating along the magnetic field H_0 are circularly polarised, and the direction of rotation of the polarisation vector is determined by the sign of the frequency ω. Instability may therefore arise for either sign of the anisotropy $1 - T_\perp/T_\parallel$, with the excitation of either right- or left-hand polarised waves, depending on the sign of $1 - T_\perp/T_\parallel$.

It follows from (3.4) that instability occurs even for very small temperature anisotropy $|T_\perp - T_\parallel| \ll T$; but the growth rate ω_i is then exponentially small.

Anisotropic plasma may be unstable even in the absence

of an external magnetic field. Consider, for example, small oscillations in a plasma with an axially symmetric (but non-isotropic) electron distribution function for which $\langle v_\perp^2\rangle/2 = (\langle v_x^2\rangle + \langle v_y^2\rangle)/2 > \langle v_\parallel^2\rangle$. For waves propagating along the axis of symmetry we have from the kinetic equation

$$f = -\frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H}\right)\frac{\partial f_0}{\partial \mathbf{v}}\frac{1}{-i(\omega - \mathbf{k}\mathbf{v})}$$

where $\mathbf{E} = (E_x, E_y, 0)$, $\mathbf{H} = (H_x, H_y, 0)$ and $\mathbf{k} = (0, 0, k)$. Substituting this expression into

$$-kH = \frac{4\pi}{c} j - \frac{i\omega}{c} E$$

where $H = H_x + iH_y$, $E = E_x + iE_y$ and $j = \int (v_x + iv_y) f \, d\mathbf{v}$, and taking into account the fact that $-kE = \frac{i\omega}{c} H$, we obtain the dispersion relation

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \cdot \left(1 - \int \frac{\partial f}{\partial v_z} \frac{dv_z}{1 - \frac{kv_z}{\omega}} \frac{k\langle v_\perp^2\rangle/2}{\omega}\right)$$

It follows from this equation that for waves with phase velocity considerably smaller than the electron thermal velocity

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{\langle v_\perp^2/2\rangle}{\langle v_\parallel^2\rangle}\right)$$

so that, as $k \to 0$, the oscillations grow with

$$\gamma \sim \omega_p\left(\frac{\langle v_\perp^2/2\rangle}{\langle v_\parallel^2\rangle} - 1\right)^{1/2}$$

THE GROWTH OF OSCILLATIONS IN PLASMA IN THE PRESENCE OF A RELATIVE MOTION OF IONS AND ELECTRONS

Plasma may be unstable even when both electrons and ions have Maxwellian velocity distributions but are in relative motion with velocity u, i.e. there is a net current. Instability should arise when u exceeds a certain critical value which is of the order of the phase velocity of the corresponding wave.

Let us consider, for simplicity, the case of zero magnetic field (the analysis can readily be extended to the case where there is a constant magnetic field and the oscillations take place along the lines of force). Longitudinal ion oscillations (ion sound waves) have the lowest phase velocity. However, if ω/k is not much greater than the average thermal velocity of the ions $\sqrt{T_i/M}$, these oscillations are rapidly damped out (in a few periods), transferring their energy to ions moving at approximately the phase velocity of the wave. These oscillations will exist only if $ZT_e \gg T_i$. This condition is often fulfilled in practice even for plasmas with $Z \approx 1$ (Z is the ion charge).

We shall find the instability criterion for ion sound oscillations by considering the interaction of particles with the potential 'barriers' of the ion sound waves. Suppose that in the coordinate system in which the average ion velocity is zero, the ion and electron distribution functions are of the form

$$f_i = (2\pi T_i / M)^{-\frac{1}{2}} \exp - Mv^2 / 2T_i$$

$$f_e = (2\pi T_e / m)^{-\frac{1}{2}} \exp - mv^2 / 2T_e$$

where f_i and f_e are total distribution functions, integrated over the transverse velocities. The energy balance between the particles and the wave is determined by the rate of transfer of energy from electrons to the wave

$$\frac{d\mathscr{E}_e}{dt} \sim \varphi_0^2 \omega_0^2 \omega \left(\frac{df_e}{dv}\right)_{v=\omega/k}$$

and by the rate of transfer of energy from the wave to the ions

$$\frac{d\mathscr{E}_i}{dt} \sim \varphi_0^2 \Omega_0^2 \omega \left(\frac{df_i}{dv}\right)_{v=\omega/k}$$

The instability criterion for the ion sound waves is then of the form

$$\frac{d\mathscr{E}}{dt} = \frac{d\mathscr{E}_e}{dt} - \frac{d\mathscr{E}_i}{dt} > 0$$

i.e.

$$\frac{u - \omega / k}{\omega / k} > \sqrt{\frac{M}{m}} \left(\frac{T_e}{T_i}\right)^{\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\frac{T_e}{T_i} + 3\right)\right] \qquad (3.5)$$

Using (3.5), we can determine the growth rate $\gamma = \frac{1}{2\mathscr{E}}\frac{d\mathscr{E}}{dt}$ by dividing the rate of energy transfer from the particles to the wave by the wave energy density $\mathscr{E} \sim \frac{n}{M}\frac{e^2\varphi_0^2}{(\omega/k)^2}$:

$$\gamma \sim \omega\left[\frac{u - \frac{\omega}{k}}{\frac{\omega}{k}}\sqrt{\frac{m}{M}} - \left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{1}{2}\left(\frac{T_e}{T_i}+3\right)}\right]$$

The frequency of ion oscillations has an upper limit which is of the order of the Langmuir frequency $\Omega_0 = \sqrt{4\pi ne^2/M}$. The phase velocity ω/k remains approximately equal to $\sqrt{2(T_e + T_i)/M}$ over the whole spectral range. It follows that γ increases linearly with frequency, and the oscillations most rapidly excited are short waves whose wavelength is of the order of a few Debye radii for ions. Measurements of the damping rate for ion sound waves with $u \approx 0$ were carried out in [3]. The results of the experiment are in satisfactory agreement with the theory.

In order for a current in a weakly ionised plasma to excite ion sound waves, it is necessary that the growth rate γ arising from the interaction between the ion sound waves and the electrons be larger than the damping rate owing to the transfer of energy from the electrons performing oscillatory motion in the wave to the atoms or molecules in the neutral gas. The condition for excitation of ion sound waves in weakly ionised plasma is thus of the form:

$$\left(\frac{\pi}{m}\frac{\partial f_e}{\partial v}\right)_{v=\omega/k} > \left(\frac{2}{M}\frac{\nu_i}{v^3 k}\right)_{v=\omega/k} \tag{3.6}$$

Results of experiments on the excitation of ion sound waves in the positive column of a glow discharge [4] are in agreement with (3.6).

If the ions and electrons are in relative motion in a magnetic field, there may be electrostatic oscillations with frequency approaching the ion cyclotron frequency Ω_i and wave vector almost perpendicular to the external magnetic field [5].

The dispersion relation for sufficiently long waves in plasma in which electrons are moving relative to ions with a drift velocity v_D is of the form [5].

$$-1+i\sqrt{\frac{\pi}{2}}\,\frac{-\omega+k_{\|}v_D}{kv_e}+\frac{T_e}{T_i}\Gamma_1(k_{\perp}^2\rho_i^2)\left\{W\left(\frac{-\omega+\Omega_i}{k_{\|}v_i}\right)\right.$$
$$\left.+\frac{\Omega_i}{\omega-\Omega_i}\left[1+W\left(\frac{-\omega+\Omega_i}{k_{\|}v_i}\right)\right]\right\}$$

where $\Gamma_1(x)=e^{-x}I_1(x)$, $I_1(x)$ is the Bessel function of imaginary argument, v_i, v_e are the ion and electron thermal velocities, $\rho_i=\frac{v_i}{\Omega_i}$ and

$$W(x)=-1+\frac{x}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{y^2/2}\frac{dy}{x+y}$$

It follows from this equation that the critical velocity v_D is approximately equal to $\left(14\frac{T_i}{T_e}+3\right)v_i$, and the frequency of oscillations arising at the limit of instability is $\sim\Omega$. These theoretical results are in agreement with experimental data on the excitation of waves by a longitudinal (relative to the magnetic field) current in potassium and caesium plasmas [6].

BEAM INSTABILITY IN A PLASMA

Since the mean free path of plasma particles is often much greater than the dimensions of the apparatus, groups of particles with different average velocities (beams) may exist in the same region in the plasma. Such beams may be introduced artificially into the plasma (e.g. electron beams in electron accelerators and ion beams injected into a trap), or they may be produced by external fields (e.g. 'runaway' electrons). In some cases the plasma itself may be considered as a number of interpenetrating electron and ion beams. In this section we shall be interested mainly in nearly mono-energetic beams in which the velocity spread is much smaller than the average velocity.

An interesting property of nearly mono-energetic beams is their rapid energy loss, which cannot be explained by the theory of binary collisions but agrees well with the theory of beam instability.

We shall confine our attention to homogeneous, quasi-neutral, infinitely extended beams, and will neglect effects due to the finite transverse dimensions of real beams.

Nearly mono-energetic beams are satisfactorily described by the hydrodynamic approximation; this is the only treatment we shall use in this discussion.

Let us consider several beams, moving with velocities V_i and composed of particles with mass m_i and charge e_i. The temperature of the particles in a beam will be denoted by T_i and the numerical density by N_i. Euler's equation for small perturbations has the form

$$\frac{\partial v_i}{\partial t} + V_i \frac{\partial v_i}{\partial x} = - \frac{1}{mN_i} \frac{\partial p_i}{\partial x} + \frac{e_i}{m_i} E \tag{3.7}$$

the continuity equation is

$$\frac{\partial n_i}{\partial t} + V_i \frac{\partial n_i}{\partial x} + N_i \frac{\partial v_i}{\partial x} = 0 \tag{3.8}$$

and the equation for the longitudinal electric field is

$$\frac{\partial E}{\partial x} = 4\pi e \sum e_i n_i \tag{3.9}$$

In these equations v_i, n_i and p_i are the perturbations of velocity, density and pressure of the particles in the i-th beam, and E is the electric field due to the perturbation. For an ideal gas of particles in each beam

$$\frac{p_i}{m_i} = \gamma_i \frac{T_i}{m_i} n_i \tag{3.10}$$

where $\gamma_i \sim 1$ is the effective adiabatic exponent (defined in rigorous kinetic theory).

For a perturbation of the form $e^{-i\omega t + ikx}$, we find the relation between ω and k by substituting (3.7), (3.8) and (3.10) into (3.9):

$$k^2 = \sum \frac{\omega_i^2}{(\omega / k - V_i)^2 - c_i^2} \tag{3.11}$$

where

$$\omega_i^2 = \frac{4\pi N_i e^2}{m_i}, \quad c_i^2 = \gamma_i \frac{T_i}{m_i}$$

Two-beam instability

Consider two beams with charges e_1 and e_2, masses m_1 and m_2, densities N_1 and N_2 and velocity spreads c_1 and c_2. The first beam is at rest, and the second moves with velocity V. Equation (3.11) now has the form

$$F\left(\frac{\omega}{k}\right) \equiv \frac{\omega_1^2}{\left(\frac{\omega}{k}\right)^2 - c_1^2} + \frac{\omega_2^2}{\left(\frac{\omega}{k} - V\right)^2 - c_2^2} = k^2 \qquad (3.12)$$

where

$$\omega_{1,2}^2 = \frac{4\pi e_{1,2}^2 N_{1,2}}{m_{1,2}}$$

If $c_1 + c_2 > V$, the beams are stable. Only long-wavelength perturbations lead to instability for $c_1 + c_2 < V$. It follows from (3.12) that, for mono-energetic beams

$$k_{\rm cr}^2 = \frac{\omega_1^2}{V^2}\left(1 + \left(\frac{\omega_2^2}{\omega_1^2}\right)^{1/3}\right)^3$$

Of the four possible waves, two do not grow. These two waves are analogous to normal Langmuir oscillations. The other two waves do not grow if they are sufficiently long.

Let us consider in more detail the origin of 'electrostatic' instability (instability with respect to perturbations in the form of longitudinal plasma oscillations is often called 'electrostatic'). As an example, let us consider two very different beams: an electron beam moving through an ion gas at rest and a dense beam moving through a tenuous one. Let us assume that a long-wavelength perturbation is initially produced in the plasma ($k \ll \omega_2/V$, where ω_2 is the electron plasma frequency). Since we are not interested in rapid oscillations, this initial perturbation should be quasi-neutral. Suppose, however, that it has a small excess negative charge, so that ions will collect in the field of this charge. On the other hand, the electrons moving in it will be retarded, and their density will also increase in the region of the negative space charge. The initial quasi-neutral perturbation will thus grow, and the characteristic growth time may be found as follows: since the energy of the moving electrons must be conserved,

$$Vv_e - \frac{e}{m}\varphi = \text{const}$$

where the constant may be set equal to zero. Here v_e is the electron velocity perturbation and φ is the potential. It follows from the conservation of electron current that

$$n_e = \frac{N_e v_e}{V} = -\frac{N_e e}{mV^2}\varphi$$

where n_e is the electron density perturbation. Poisson's equation then gives the following expression for the ion density perturbation

$$n_i = \frac{1}{4\pi e}\left(-\frac{\partial^2\varphi}{\partial x^2} - \frac{4\pi e^2 N}{mV^2}\varphi\right) \tag{3.13}$$

It follows from this that for sufficiently long waves, as we have already seen above, both the ion and the electron density will increase in the region of the negative space charge. From the equation of motion for ions in the field $-\partial\varphi/\partial x$, and from the continuity equation, we have

$$\frac{\partial v_i}{\partial t} = -\frac{e}{M}\frac{\partial\varphi}{\partial x}, \quad \frac{\partial n_i}{\partial t} = -N_i\frac{\partial v_i}{\partial x}$$

i.e.

$$\frac{\partial^2 n_i}{\partial t^2} = \frac{N_i e}{M}\frac{\partial^2\varphi}{\partial x} \tag{3.14}$$

For a perturbation which is sinusoidal in space we have from (3.13) and (3.14)

$$\frac{\partial^2 n_i}{\partial t^2} = \frac{4\pi e^2 N_i}{M}\frac{k^2}{k^2 - \frac{4\pi e^2 N}{mV^2}} n_i$$

The characteristic growth time is

$$\frac{1}{\gamma} = \left(\frac{\frac{4\pi e^2 N}{mV^2} - k^2}{\frac{4\pi e^2 N}{M} k^2}\right)^{1/2} \approx \frac{1}{kV}\left(\frac{M}{m}\right)^{1/2} \tag{3.15}$$

The closer the wavelength approaches the 'resonance' value ($k_{res} = \omega_e/V$), the faster the perturbation grows.

The minimum growth time, i.e. growth at resonance, can be found from (3.12) by substituting $k = \omega_e/V$ and neglecting ω/ω_e in comparison with unity:

$$\frac{1}{\gamma} = \left(\frac{2M}{m}\right)^{1/3}\frac{2}{\omega_e\sqrt{3}} \tag{3.16}$$

where γ is the imaginary part of ω (the growth rate).

Stability of ion beams in plasma

Let us consider an ion-electron plasma in which there is an ion beam. We shall assume for simplicity that the ion temperature is much lower than the electron temperature, and introduce the following notation: ω_{p1} is the Langmuir frequency of the plasma ions (first beam), ω_{p2} is the Langmuir frequency of the beam ions (second beam), ω_e is the electron Langmuir frequency, V is the velocity of the second ion beam, k is the wave number of the perturbation and $c_e = \sqrt{\gamma(T/m)}$ is the electron thermal velocity.

The dispersion equation (3.11) is now of the form

$$F\left(\frac{\omega}{x}\right) = \frac{\omega_{p1}^2}{(\omega/k)^2} + \frac{\omega_{p2}^2}{(\omega/k - V)^2} + \frac{\omega_e^2}{(\omega/k)^2 - c_e^2} = k^2 \tag{3.17}$$

Stability or instability of the beams depends on whether or not the function $F(\omega/k)$ has a zero in the interval $0 < \omega/k < V$ The beams are stable if the equation $F = 0$ has two real roots, and are unstable if it has four. Consider the special case $V \ll c_e$. The ratio ω/k can then be neglected in (3.17) in comparison with c_e, and we obtain

$$\frac{\omega_{p1}^2}{(\omega/k)^2} + \frac{\omega_{p2}^2}{(\omega/k - V)^2} = \frac{\omega_e^2}{c_e^2} \tag{3.18}$$

This equation is equivalent to (3.12). From (3.18) we thus obtain the stability criterion

$$\frac{V^2}{c_e^2} > \frac{\omega_{p2}^2}{\omega_e^2}\left(1 + \left(\frac{\omega_{p_1}}{\omega_{p_2}}\right)^{2/3}\right)^3$$

On the other hand, our discussion is valid only for $V^2 < c_e^2$; violation of this condition leads to electrostatic instability on the electron branch. The stability criterion is therefore of the form

$$1 > \frac{V^2}{c_e^2} > \frac{\omega_{p2}^2}{\omega_e^2}\left(1 + \left(\frac{\omega_{p_1}}{\omega_{p_2}}\right)^{2/3}\right)^3 \tag{3.19}$$

We have considered only a special type of perturbation,

i.e. oscillations along the direction of the beam velocity. Ion beams are always unstable with respect to the growth of 'oblique' perturbations ($\mathbf{k} \nparallel \mathbf{V}$). However, if there is a sufficiently strong magnetic field in the plasma, which is parallel to the beam velocity, the criterion given by (3.19) will remain valid, i.e. sufficiently rapid ion beams are stable. This is related to the fact that a magnetic field 'inhibits' motion perpendicular to **H**, so that the wave-vector component $k_\perp$ drops out of the equations. This result is confirmed by experiments on collisions between plasma bunches moving in the direction of a magnetic field.

Effect of a magnetic field on beam instability

The presence of a magnetic field perpendicular to the direction of the beam may alter the results of the previous section. A magnetic field influences the motion of ions in a wave if $\lambda \gtrsim R_i$, where R_i is the ion Larmor radius and λ is the 'resonance' wavelength $\lambda = \frac{2\pi}{k} = 2\pi \frac{V}{\omega_{pi}}$. Since $R_i = \frac{v_{Ti}}{\omega_{Hi}}$ (where v_{Ti} is the thermal velocity and ω_{Hi} is the cyclotron frequency of the ions), a magnetic field will influence ion motion in perturbations only if

$$\frac{V}{v_{Ti}} > \frac{\omega_{pi}}{\omega_{Hi}} = \left(\frac{4\pi N Mc^2}{H^2}\right)^{1/2} \tag{3.20}$$

Consider a beam in a sufficiently weak field, i.e. suppose that (3.20) is not fulfilled. If the 'resonance' wavelength of the perturbation is greater than the electron Larmor radius, the field will have an appreciable influence. Thus for $\frac{V}{c_e} > \frac{\omega_{pi}}{\omega_{Hi}}$ the electrons are 'magnetised', i.e. they drift under the influence of the magnetic field of the perturbation wave.

The perturbation of the x-component of the electron velocity (the beam moves along the x axis) is therefore given by

$$v_{ex} = -\frac{mc^2}{eH^2}\frac{\partial E_x}{\partial t}$$

From the continuity equation for electrons, it follows that

$$\frac{\partial n_e}{\partial t} = -N_e \frac{\partial v_{ex}}{\partial x} = \frac{mc^2 N_e}{c^2 H^2} \frac{\partial^2 E_x}{\partial t \partial x}$$

i.e.

$$n_e = \frac{mc^2 N_e}{eH^2} \frac{\partial E_x}{\partial x} \tag{3.21}$$

The electric field in a longitudinal wave is defined by

$$\frac{\partial E_x}{\partial x} = +4\pi e\,(n_i - n_e)$$

Substituting into this for n_i from (3.7) and (3.8) and for n_e from (3.21), and neglecting the left-hand side, we obtain

$$\frac{4\pi N mc^2}{H^2} k^2 = \frac{\omega_{i1}^2}{(\omega/k)^2 - v_{Ti1}^2} + \frac{\omega_{i2}^2}{(\omega/k - V)^2 - v_{Ti2}^2} \tag{3.22}$$

This equation coincides in form with (3.12). The stability criterion is therefore again of the form $V > v_{Ti\,1} + v_{Ti\,2}$ and the resonance wavelength is given by

$$k_{\text{res}} = \frac{\omega_{i2}\omega_H}{\omega_e V} \approx \frac{\sqrt{\omega_{Hi}\omega_{He}}}{V}$$

i.e. resonance occurs at the 'geometric mean' of the frequencies, $\sqrt{\omega_{Hi}\omega_{He}}$.

Equation (3.22) will not be valid when the beam velocity V, being greater than the ion thermal velocity, nears the Alfvén velocity $H/\sqrt{4\pi\rho}$. For high velocities the beam is unstable with respect to oblique perturbations $\mathbf{k} \nparallel \mathbf{V}$.

STABILITY OF PLASMA BOUNDARY

From the point of view of applications, e.g. controlled thermonuclear fusion, one of the most interesting questions is that of the stability of plasma containment by a magnetic field, i.e. the stability of a magnetically isolated plasma. Consider a layer of plasma in a vacuum. The boundary is kept in equilibrium by the magnetic field pressure $H^2/8\pi$. At the ends, the plasma is bounded by perfectly conducting plates which are perpendicular to the magnetic field and separated by a distance L. The fields within and around the plasma are parallel to the plasma boundary and the force $\mathbf{f} = \rho \mathbf{g}$ acts at right angles to it. The fields inside and

outside the plasma are B_0 and H_0 respectively.

If the plasma boundary is displaced vertically by δz, and the perturbation has a length of the order of l across the field, then the pressure on the most deflected part of the boundary increases by the weight of a column of plasma of height δz:

$$\delta p = \rho g \delta z$$

The distortion of the magnetic field gives rise to a quasi-elastic force, and if this force is greater than the change in pressure, the equilibrium is stable. In a vacuum, this force is connected with the change in the magnetic pressure $\delta H^2/8\pi = H_0 \delta_{\parallel}/4\pi$, where $\delta H_{\parallel}$ is the component of the perturbation of the magnetic field along H_0. The field along the boundary of perfectly conducting plasma remains parallel to it. It follows that $\delta H_{\perp} \sim H_0 \delta z/L$. Since the motion is quasi-stationary $(v/c \rightarrow 0)$, outside the plasma $\mathbf{H} = -\nabla\Psi$ and $\Delta\Psi = 0$. For $l \ll L$: the characteristic linear dimension of the field perturbation in the direction perpendicular to the boundary is therefore equal to l. Consequently $\delta H_{\parallel} \sim \delta H_{\perp} l/L$ and $\delta H^2/8\pi \sim H_0^2 \, l \delta z/L^2$. Inside the plasma the field is 'frozen' and the body force is the same as the Maxwellian tension $B_0^2/4\pi R$, where R is the radius of curvature of the lines of force. Since $R \sim L^2/\delta z$, the quasi-elastic body force is equal to $B_0^2 \delta z/L^2$. The equilibrium is stable if

$$\frac{B_0^2 + H_0^2}{L^2} > \frac{g\rho}{l} \tag{3.23}$$

For any field there will be perturbations for which l will be so small that a sharp plasma boundary will be unstable with respect to them. It is clear that such perturbations diffuse the boundary to a thickness

$$l_b \sim g\rho L^2/B_0^2$$

In reality, both a quasi-elastic force $B_0^2 \delta z/L^2$ and a hydrostatic force $g\delta\rho$ arise if an element of an inhomogeneous plasma is displaced. Here $\delta\rho$ is the density difference between the displaced element and the surrounding plasma. However

$$\delta\rho = \nabla \rho_0 \cdot \delta z,$$

so that the stability criterion is

$$\frac{B^2}{L^2} > g\nabla\rho_0 \sim \frac{g\rho}{l_b} \tag{3.24}$$

i.e. a diffuse plasma boundary of width l_b is stable.

When dense plasma does not occupy all the space between the end plates, but only a length $L_1 \ll L$, and there is a region of tenuous plasma near the plates, we must substitute the 'effective' density $\rho^* = \rho(L_1/L)$ for ρ in (3.24). The exact necessary and sufficient stability criterion is then of the form

$$\frac{\pi}{4}\frac{B_0^2}{L_1 L\rho} > \frac{g}{l} \tag{3.25}$$

In reality, the system remains unstable (but the growth rates are smaller) because of the finite conductivity of the plate material and, particularly, because of the tenuous secondary plasma through which contact is made with the end plates. Although we have used the hydrodynamic description, these results hold qualitatively even for a tenuous plasma.

It is readily seen that in the absence of a stabilising force connected with the fixed ends, i.e. for $L \to \infty$, the hydrodynamic instability described above will be characterised by a growth rate $\gamma \sim \sqrt{g/l}$, since a pressure change $\delta p = \rho g \delta z$ will then lead to an acceleration

$$\gamma^2 \delta z \sim \delta\ddot{z} = \frac{\delta p}{\rho l} = \frac{g}{l}\delta z \tag{3.26}$$

Any force which acts at right angles to the magnetic field and whose action does not depend on the sign of the charge will give rise to the above instability. Here, as in the case of the gravitational field, the conducting ends have a stabilising influence. The centrifugal force associated with the motion of particles along a curved line of force may act in this way. We must then substitute $\mathrm{R}\frac{v_\parallel^2}{R^2}$ for g in (3.24), where R is the radius of curvature of the line of force. Secondly, such a force may be connected with drift in a non-uniform magnetic field, in which case $g \to \frac{\mathrm{R}}{R^2}\frac{v_\perp^2}{2}$. Combining these two effects, we obtain

$$g \to \frac{\mathrm{R}}{R^2}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right) \sim \frac{\mathrm{R}}{R^2}\frac{p_\parallel + p_\perp}{\rho} \tag{3.27}$$

It is clear from this that a convex plasma boundary is unstable. Moreover, (3.25) and (3.27) show that if there are conducting plates at the ends of the system, a necessary condition for stabilisation is

$$B^2 > \frac{4L^2\rho}{l_b} \frac{v_\parallel^2 + v_\perp^2/2}{R} \tag{3.28}$$

where L is the average distance between the plates, obtained by taking into account the variation of the plasma density along the line of force, R is the radius of curvature of the line of force and l_b is the thickness of the diffuse boundary.

Consider a cylinder of plasma placed in crossed electric and magnetic fields, i.e. rotating with velocity $v = cE/H$. In this case the centrifugal force may lead to instability. We have

$$g_{eff} \sim \frac{v^2}{r} \sim \frac{c^2E^2}{rH^2} \tag{3.29}$$

where r is the radius of the plasma boundary. It follows from (3.29) that

$$\gamma \sim \sqrt{g_{eff}/l} \sim \frac{cE}{H\sqrt{rl}}$$

When the plasma is bounded by an inhomogeneous magnetic field in a vacuum, its stability depends on whether the field increases or decreases with distance from the plasma. If there are several perturbations at the plasma boundary in the form of 'tongues' protruding between the lines of force, we may apply arguments already discussed to study the subsequent behaviour of the system. The pressure at the tip of a slow-moving tongue is the same as the plasma pressure, i.e. $H_b^2/8\pi$, where H_b is the magnetic field at the boundary. The surrounding magnetic pressure is $H^2/8\pi$, where H is the amplitude of the magnetic field at a distance δz from the boundary, i.e.

$$\delta p = \frac{H^2(\delta z) - H^2(0)}{8\pi} = \frac{H_b}{4\pi} \frac{dH}{dz} \delta z$$

and from (3.26) we obtain

$$\gamma \sim \sqrt{H\nabla H/4\pi\rho l}$$

This result is significant even for high-frequency fields. The stability criteria, taking into account the effect of 'fixed ends' of the lines of force, can readily be obtained by analogy with (3.24).

When a plasma undergoes acceleration, the analogue of $\mathbf{g}$ is the acceleration $\mathbf{a}$ of the boundary. The growth rate of the perturbation is then $\gamma \sim \sqrt{a/l}$. Such instabilities are observed in experiments on plasma constricted by an axial magnetic field (theta pinch).

The viscosity of plasma has a stabilising influence on the boundary with respect to the growth of flute perturbations [7]. Consider the oscillations in non-uniform plasma (placed in a magnetic field $\mathbf{H}_0$ and a gravitational field $\mathbf{g}$ where $\mathbf{g}\cdot\mathbf{H}_0 = 0$) which do not curve the magnetic lines of force (two-dimensional motion). For low-pressure plasmas ($nT \ll H_0^2/8\pi$) we can use the equations

$$\begin{aligned} i\omega\rho + \nabla\rho_0\mathbf{v} &= 0 \\ i\omega\rho_0\mathbf{v} &= -\nabla\left(p + \frac{\mathbf{H}\cdot\mathbf{H}_0}{4\pi}\right) - \nabla\pi + \rho\mathbf{g} \end{aligned} \tag{3.30}$$

where ρ is the plasma density, $\mathbf{v}$ is its velocity, $\mathbf{H}$ is the perturbation of the magnetic field and the viscous stress tensor π is given by ($\omega_H \gg \nu$) [8]

$$\pi = \pi^S + \pi^H$$

$$\pi^S_{xx} = \pi^S_{yy} = -\eta_S \frac{1}{2}(W_{xx} + W_{yy}), \quad \eta_S \approx \frac{nT}{\nu}$$

$$W_{ik} = \nabla_k v_l + \nabla_i v_k - \frac{3}{2}\delta_{ik}\nabla_e v_e$$

$$\pi^H_{xx} = -\pi^H_{yy} = -\eta_H W_{xy}; \quad \pi^H_{xy} = -\pi^H_{yx} = \eta_H \frac{1}{2}(W_{xx} - W_{yy})$$

$$\eta_H = \frac{nT}{2\omega_H}$$

The z axis of the rectangular coordinate system lies along $\mathbf{H}_0$ and the x axis along $\mathbf{g}$. Eliminating the pressure $p + \mathbf{H}\cdot\mathbf{H}_0/4\pi$ from (3.30) and noting that for flute perturbations $\nabla\cdot\mathbf{v} = 0$, we obtain a dispersion relation for the frequency and the projection of the wave vector along the y axis:

$$\omega^2 - \frac{ck(nT_i)'}{eH_0n}\omega + g\frac{n'}{n} = 0 \tag{3.31}$$

It follows from (3.31) that a plasma boundary is stable with respect to flute perturbations if their wavelength is less than the critical value

$$\lambda_{cr} = \pi \frac{cT_i}{eH_0} \frac{n'}{n} \Big/ \left(g \frac{n'}{n}\right)^{1/2}$$

Experimental studies of flute instabilities in magnetic mirror traps were reported in [9]. It was shown that when the main magnetic field is supplemented by the magnetic field of linear current-carrying conductors parallel to the trap axis, the plasma is stabilised, as predicted by theory (the curvature of the lines of force near the linear conductors is such that the effective acceleration g is in the direction from the vacuum to the plasma), so that there is no instability.

CONVECTIVE INSTABILITY

Let us now consider the stability of the internal portions of a plasma with closed lines of force. For simplicity we shall confine our attention to axially symmetric plasma configurations. Let us suppose that the field in the plasma has only an azimuthal component B_0, i.e. current flows in the plasma along the axis, and that the current density, the magnetic field and the plasma pressure and density depend only on the distance from this axis.

Because of the tension of the curved magnetic field lines of force, each tube of force tends to contract towards the axis. This is opposed by the magnetic and gas pressures. In the stationary state these two forces are in equilibrium. If the tension in a tube increases more rapidly than the pressure gradient when the tube is displaced through a small distance towards the axis, then the equilibrium is unstable and vice versa.

To investigate plasma stability, we must calculate the total force acting on a displaced tube. Consider a narrow tube of force at a distance r from the axis. The gradient of the total pressure is balanced by the tension of the magnetic lines of force in the tube

$$\frac{\partial}{\partial r}\left(p + \frac{B_0^2}{8\pi}\right) = -\frac{B_0^2}{4\pi r_0}$$

Let us find the force acting on a tube displaced through a

distance δr. Since the total magnetic flux through the cross-section S_T of the tube is conserved, the change in the field B_T in the tube due to this displacement is given by

$$\frac{\delta B_T}{B_T} = -\frac{\delta S_T}{S_T} = -\frac{\delta V_T}{V_T} + \frac{\delta r}{r}$$

where V_T is the volume of the tube. However, for adiabatic motion $\delta V_T/V_T = -\delta p/\gamma p$ and therefore

$$\frac{\delta B_T}{B_T} = \frac{\delta p}{\gamma p} + \frac{\delta r}{r} \tag{3.32}$$

On the other hand, the total pressure in the tube should be equal to the external pressure at the point $r + \delta r$:

$$p_T + B_T^2/8\pi = p_0(r + \delta r) + B_0^2(r + \delta r)/8\pi$$

and so

$$\delta p + \frac{B_0 \delta B_T}{4\pi} = (p_0' + B_0 B_0'/4\pi)\,\delta r$$

Substituing for δp from (3.32), we obtain

$$\frac{\delta B_T}{B_0} = \frac{1}{\gamma p_0}\,\frac{p' + B_0(B_0' - B_0/r)/4\pi}{1 + B_0^2/4\pi\gamma p}\,\delta r + \frac{\delta r}{r} \tag{3.33}$$

The change in tension in the magnetic tube is

$$\delta\left(\frac{B_T^2}{4\pi r}\right) = \frac{B_0 \delta B_T}{2\pi r} - \frac{B_0^2}{4\pi r^2}\,\delta r$$

Since the gradient of the total pressure at the point $r + \delta r$ is in equilibrium with the tension of tubes at that distance, the change in the total gradient during the displacement is

$$\frac{B_0^2(r + \delta r)}{4\pi(r + \delta r)} - \frac{B_0^2(r)}{4\pi r} = \frac{B_0 B_0'}{2\pi r}\,\delta r - \frac{B_0^2}{4\pi r^2}\,\delta r$$

It follows that the total force acting on a tube is

$$\delta F = \delta\left(\frac{B_T^2}{4\pi r}\right) - \delta\left(\frac{B_0^2}{4\pi r}\right) = \frac{B_0}{2\pi r}\,(\delta B_T - B_0'\delta r)$$

Substituting (3.33), we find that

$$\delta F = -\frac{B_0^2}{2\pi r}\left\{\frac{1}{\gamma p_0}\,\frac{p_0' + B_0(B_0' - B_0/r)/4\pi}{1 + B_0^2/4\pi\gamma p_0} + \frac{1}{r} - \frac{B_0'}{B_0}\right\}\delta r$$

If $\delta F/\delta r > 0$, the tube is accelerated and the equilibrium is unstable. The stability criterion for the rearrangement of tubes of force is therefore of the form

$$\delta F/\delta r < 0$$

Since in equilibrium

$$p_0' = \frac{B_0}{4\pi}(B_0' + B_0/r)$$

the plasma is stable if

$$\frac{B_0'}{B_0} - \frac{1}{r} + \frac{B_0^2}{4\pi\gamma p_0}\left(\frac{B_0'}{B_0} + \frac{1}{r}\right) < 0 \tag{3.34}$$

For an incompressible fluid ($\gamma \to \infty$) the necessary and sufficient condition for stability is that the field does not increase more rapidly than the distance from the axis. For a tenuous plasma ($\beta = \frac{8\pi p}{B_0^2} \ll 1$) to be stable, the field must fall off more rapidly than $1/r$.

Substituting $U = -\int dl/B_0 = -\frac{2\pi r}{B_0}$ and using (3.34) we have for $\beta \ll 1$

$$\nabla U \nabla p < \gamma p \frac{(\nabla U)^2}{|\nabla U|}$$

This condition is valid for any shape of closed lines of force [2].

STABILITY OF A PLASMA COLUMN

A plasma column compressed by its own current is unstable with respect to constrictions and kinks. A strong axial magnetic field may be used to stabilise the column. The stabilisation effect arises from work being expended to increase the energy of the magnetic field of the column when it is deformed. This effect is most clearly defined for perturbations with a large wave vector along the column axis. On the other hand, in long-wavelength perturbations the change in the longitudinal magnetic field is small, and the column remains unstable with respect to perturbations whose wavelength is considerably greater than the radius of the column. The stability criterion for a column with respect to constrictions and kinks can be obtained as follows.

Let us consider first kink-type perturbations of a pinch.

We shall assume that there is a trapped axial field B inside the column of radius a, and an azimuthal field H outside, due to a current flowing on its surface. If the column curves over a distance λ, the density of the lines of force of the azimuthal field will be increased on one side of the column and reduced on the other. A large magnetic pressure will thus act on the side of the column facing the centre of curvature. On the other hand, the curvature of the lines of force of the trapped field will give rise to a force acting in the other direction.

The force due to the azimuthal field acting on a unit length of the column may be calculated as follows. Consider a cylindrical volume of radius λ surrounding the column and let us cut this cylinder by planes passing through the centre of curvature. Since the lines of force of the azimuthal field lie on these planes, the total force acting in the direction of the displacement consists of the corresponding component of the magnetic pressure at the end faces.

$$2\int_0^{\lambda} \frac{H^2}{8\pi}\, 2\pi r dr \alpha$$

(where $\alpha = \lambda/2R$, R being the radius of curvature) and the pressure on the lateral surfaces, which may be neglected. The perturbation vanishes at distances $\sim\lambda$.

The force per unit length of the column arising from the perturbation of the field is

$$\frac{1}{R}\int_a^{\lambda} \frac{H^2}{8\pi}\, 2\pi r dr = \frac{H_0^2}{4R} \ln\frac{\lambda}{a}\, a^2$$

where $H_0 = H(a)$ is the field at the surface of the column.

The quasi-elastic body force is

$$-\frac{B^2}{4\pi R}\pi a^2 = \frac{B^2}{4R} a^2$$

so that the total force is

$$\delta F = \frac{a^2}{4R}\left(H_0^2 \ln\frac{\lambda}{a} - B^2\right)$$

Since in equilibrium

$$p + \frac{B^2}{8\pi} = \frac{H^2}{8\pi}$$

it follows that $H^2 > B^2$ and it is clear that the column cannot be stabilised with respect to long-wavelength perturbations by a strong internal longitudinal field.

If there is a longitudinal magnetic field both inside and outside the column carrying an axial current, the resultant field will be helical. The column will then curve along a helical line of force and may lie between the lines of force without distorting it. This type of instability will occur if the perturbation of the surface of the column is helical and if the pitch λ of the helix is equal to or greater than the pitch $2\pi a \frac{H_z}{H_\varphi}$ of the force lines on the surface of the column. It follows that the column is stable with respect to helical perturbations with wavelengths

$$\lambda < 2\pi a \frac{H_z}{H_\varphi}$$

If the perturbation wavelength is limited by the dimensions of the system (for example, the length of a toroidal pinch), a current greater than a critical value (the Shafranov-Kruskal current) will lead to instability. This current is

$$I_{\text{cr}} \sim c \frac{a^2}{R} H_z$$

Thus, in both cases there exists a maximum wavelength for a perturbation which may be stabilised by a magnetic field.

The stability criterion for the growth of a constriction can be found as follows. Let the radius of the plasma column be changed by δa. Then, since the flux is conserved, the field inside the column will change by

$$\delta B = -B_0 \frac{\delta a}{a}$$

On the other hand, the azimuthal field outside the column will be

$$H = \frac{2I}{ca}$$

where I is the total current. It follows that $\delta H = -H_\varphi \frac{\delta a}{a}$.

The total change in the difference between the internal and external magnetic pressures is then given by

$$\delta p_H = -\frac{B^2}{4\pi}\frac{2\delta a}{a} + \frac{H^2}{4\pi}\frac{\delta a}{a}$$

so that the stability criterion is

$$B^2 > \frac{H^2}{2}$$

We conclude that a sufficiently strong longitudinal magnetic field will suppress constrictions, but cannot stabilise a pinch with respect to long-wave kinks.

Experiments with high-current discharges have confirmed the results of the linear theory of stability of a plasma column. Experimental data [10] on the behaviour of a stream of liquid metal carrying a current, with and without external longitudinal magnetic fields, are in agreement with the linear theory of plasma column stability (the long-term development of perturbations growing in an unstable stream was also investigated).

In a helical field the pitch of the lines of force is different on each magnetic surface, the tubes 'entangle' under a radial displacement and a quasi-elastic force arises. However, in any given layer, a tube may shift if its displacement is uniform along the lines of force. The quasi-elastic force grows as it moves out of its own layer, so that it is possible for a surface instability to arise near a certain magnetic surface.

If a perturbation is not to distort the line of force, its wave vector must everywhere be perpendicular to the line. For a perturbation of the form $e^{im\varphi+ikz}$ (φ is the azimuthal angle), this condition may be written in the form

$$\mathbf{k}\cdot\mathbf{B} = \frac{m}{r}B_\varphi + kB_z = 0 \tag{3.35}$$

where B is the unperturbed helical field.

Let us assume that this condition is fulfilled for given m, k at some radius r_0. If the perturbation is such that $\lambda \sim \frac{1}{k} \ll r_0$, the equation of motion may be expanded near r_0, determined by (3.35), in powers of λ/r_0. Let us find the conditions under which slow displacements of the tubes are possible, i.e. the limit of stability. The equilibrium conditions for a tube are

$$\nabla\Phi + (\mathbf{B}\cdot\nabla)B/4\pi = 0, \quad \Phi = p + B^2/8\pi$$
$$\nabla\cdot\mathbf{B} = 0$$

Linearising this equation, we obtain

$$\begin{gathered}
i\mathbf{k}\cdot\mathbf{B}b_r - \frac{2B_\varphi}{r}b_\varphi - \Phi_1' = 0\\
i\mathbf{k}\cdot\mathbf{B}b_\varphi + (B_\varphi' + B_\varphi/r)\,b_r - \frac{im}{r}\Phi_1 = 0\\
i\mathbf{k}\cdot\mathbf{B}b_z - ik\Phi_1 = 0\\
\frac{1}{r}\frac{d}{dr}(rb_r) + i\mathbf{k}\cdot\mathbf{b} = 0
\end{gathered} \tag{3.36}$$

where $\mathbf{b}$ and Φ_1 are the perturbations of the field and of the total pressure.

Substituting $x = \frac{r}{r_0} - 1$ and expanding $\mathbf{k}\cdot\mathbf{B}$ in terms of x we have $\mathbf{k}\cdot\mathbf{B} = Sx$, where

$$S = r_0(\mathbf{k}\cdot\mathbf{B})'$$

From (3.36) we obtain the following differential equation for b_r

$$b_r'' + (-k^2 + \varkappa^2/x^2)\,b_r = 0 \tag{3.37}$$

where

$$\varkappa^2 = 2B_\varphi(B_\varphi' + B_\varphi/r_0)\,k^2r_0/S^2 = -\frac{8\pi p'/rB_z^2}{(\mu'/\mu)^2}$$

and $\mu = B_\varphi/rB_z$ is the torsion (i.e. the reciprocal of the radius of torsion). Generally speaking, (3.37) is invalid near $x = 0$, and for a correct description of this region it is necessary to take into account either inertia or dissipative processes. However, the formal solution of (3.37) is

$$b_r = b^0\sqrt{x}\,K_{i\nu}(kx)$$

where $\nu = \sqrt{\varkappa^2 - 1/4}$. When $\nu^2 > 0$ this function has an infinite number of zeros near $x = 0$. Solutions which damp out at infinity may therefore be matched to any solution near $x = 0$. This cannot be done for $\nu^2 < 0$. Thus, the critical value of $\varkappa^2$ is $1/4$. Rigorous consideration of inertia confirms this result.

The condition for plasma stability in a helical field is

therefore of the form $\varkappa^2 < 1/4$, i.e.

$$\frac{\Upsilon}{4}\left(\frac{\mu'}{\mu}\right)^2 + \frac{8\pi p'}{B_z^2} < 0 \tag{3.38}$$

It follows from (3.38) that, in general, the greater the relative change in the torsion of the lines of force, μ'/μ, in the radial direction, the greater the pressure gradient which may be stably maintained by the magnetic field.

Experimental data on the instability of a plasma column in a helical magnetic field are reviewed in [11].

CONVECTION

Steady convective motion (Bénard cells) arises in a heavy conducting fluid with $4\pi\sigma\chi/c^2 < 1$ (χ is the temperature diffusivity and σ the electrical conductivity) when it is heated from below in a magnetic field and the temperature gradient exceeds a certain critical value.

Let us consider a simplified model of convection, the motion of a fluid in a narrow tube of current whose dimensions along and at right angles to the field (which is directed vertically) are $\lambda_\parallel$ and $\lambda_\perp$ respectively. It is clear that the velocities of the fluid parallel and perpendicular to the magnetic field are related by

$$v_\parallel/\lambda_\parallel \sim v_\perp/\lambda_\perp$$

The work done on a unit volume in unit time by the viscous forces is

$$\eta\,|\Delta|\,(v_\parallel^2 + v_\perp^2), \quad |\Delta| = \frac{1}{\lambda_\parallel^2} + \frac{1}{\lambda_\perp^2}$$

The work done by the electromagnetic drag force is

$$j\,\frac{H}{c}\,v_1 \sim \frac{\sigma H^2}{c^2}\,v_\perp^2$$

and the work done by the hydrostatic force is

$$(\rho - \langle\rho\rangle)\,g v_\parallel = \alpha\,\langle\rho\rangle\,(T - \langle T\rangle)\,g v_\parallel \tag{3.39}$$

where α is the volume expansion coefficient of the fluid and T is the temperature averaged over all tubes at a

given height. Within a given tube

$$\Delta\,(T - \langle T\rangle) = \frac{1}{\chi}\, v_{\parallel} \nabla \langle T\rangle$$

and so

$$T - \langle T\rangle = -\frac{1}{|\Delta|}\,\frac{v_{\parallel}}{\chi}\,\frac{T_2 - T_1}{d} \tag{3.40}$$

Substituting (3.40) into (3.39) and equating the work done by the hydrostatic force to the sum of the work done by the viscous and electromagnetic drag forces, we obtain the stability criterion

$$\mathrm{Ra} < \mathrm{Ra}_{cr}$$

where $\mathrm{Ra} = -\frac{\alpha\rho g\,(T_2 - T_1)}{\chi\eta}\, d^3$ is the Rayleigh number, the critical value of which is given by

$$\mathrm{Ra}_{cr} = \frac{|\Delta|}{\lambda_{\parallel}^2}\, d^4 \left[\, |\Delta|\, \lambda_{\parallel}^2 \left(1 + \frac{\lambda_{\perp}^2}{\lambda_{\parallel}^2}\right) + \Gamma^2 \frac{\lambda_{\perp}^2}{d^2}\,\right] \tag{3.41}$$

$$\Gamma^2 = H^2 \sigma\, d/c^2 \eta$$

The ratio $d/\lambda_{\parallel}$ is determined by the boundary conditions $(d/\lambda_{\parallel} \sim \pi)$.

The form of the Bénard cells and the dependence of the critical Rayleigh number Ra_{cr} on the magnetic field may be obtained by finding the minimum of the right side of (3.41). In the model we are considering,

$$\frac{\lambda_{\perp}}{\lambda_{\parallel}} \sim \Gamma^{-1/3}$$

i.e. as the magnetic field increases, the cells expand vertically and

$$\mathrm{Ra}_{cr} \sim \frac{d^2}{\lambda_{\parallel}^2}\, \Gamma^2 \sim \pi^2 \Gamma^2, \quad \Gamma \to \infty$$

For convection between solid plates $\mathrm{Ra}_{cr} = 1{,}700$ as $\Gamma \to 0$. Accurate values of Ra_{cr} as a function of Γ are given in [12].

We have been considering the dynamics of a fluid with the equation of state $\rho = \rho_0\,[1 - \alpha(T - T_0)]$. Generalisation to the case of real plasma presents no difficulty. This can be done by replacing the temperature gradient $\frac{T_2 - T_1}{d}$ by the difference between this quantity and the adiabatic temperature gradient in the Rayleigh number

$$\mathrm{Ra} = -\left(\nabla T + \frac{g}{c_p}\right)\frac{g\rho}{T}\frac{d^2}{\chi\eta}$$

so that the analysis is valid for

$$|\nabla T| > \frac{g}{c_p}$$

where c_p is the specific heat at constant pressure. Experimental studies of thermal convection in a conducting fluid in a magnetic field agree well with theory [12].

TAYLOR VORTICES IN ROTATING PLASMA

Let us consider the stability of the steady-state rotation of a plasma with angular velocity $\Omega(r)$ between two coaxial cylinders in an axial magnetic field. Each element of plasma moves in a circle of radius r about the axis of the cylinder. The centrifugal force $\rho\Omega^2 r$ acting on each element of plasma is balanced by the total pressure gradient. For a sufficiently rapid radial displacement of the plasma element to a point $r + \delta r$, its angular momentum $\rho\Omega r^2$ is conserved, and so the force acting on it during the displacement is equal to the difference between the change in total pressure gradient and the centrifugal force.

$$\delta F = -\rho\,\frac{\delta\,(\Omega r^2)^2}{r^3} = -\frac{2\rho\Omega}{r}(\Omega r^2)'\,\delta r$$

The time for such a displacement may be evaluated by setting this force equal to the sum of the electromagnetic drag force and the viscous friction

$$-\frac{2\rho\Omega}{r}(\Omega r^2)'\,\delta r = \frac{\sigma H^2}{c^2}\frac{\delta r}{\delta t} + \frac{\eta}{\lambda^2}\frac{or}{\delta t}$$

where λ is the characteristic scale of the perturbation. Hence

$$\delta t_1 = \frac{\dfrac{\sigma H^2}{c^2} + \dfrac{\eta}{\lambda^2}}{-\dfrac{2\rho\Omega}{r}(\Omega r^2)'}$$

However, the rotational velocities of the element and of the surrounding plasma may not become equal in this time. The time for this to happen is determined approximately by

$$\rho \frac{\delta v}{\delta t_2} = \frac{\sigma H^2}{c^2} \delta v + \frac{\eta}{\lambda^2} \delta v$$

Equating δt_1 and δt_2, we find the limit of stability:

$$T_{cr} \approx \left(\frac{R_2 - R_1}{\lambda}\right)^4 \left(1 + \frac{\sigma H^2 \lambda^2}{c^2 \rho \nu}\right)^2$$

where

$$T = -\frac{2\Omega (\Omega r^2)' (R_2 - R_1)^4}{r \nu^2}$$

is the Taylor number.

For solid cylinders $T_{cr} \to T^0_{cr} = 1{,}700$ as $H \to 0$. Writing

$$\Gamma^2 = \frac{\sigma H^2}{c^3 \rho \nu} (R_2 - R_1)^2$$

we see that

$$\frac{T_{cr}}{T^0_{cr}} \approx \left(1 + \Gamma^2 \frac{\lambda^2}{(R_2 - R_1)^2}\right)^2 \frac{\lambda_0^4}{\lambda^4}$$

The length λ in all these expressions is the mean length of the perturbation

$$\lambda \approx (R_2 - R_1) A$$

and the precise value of the numerical factor A depends on the boundary conditions. In a strong field $(\Gamma \gg 1)$ the limit of stability is determined by the dimensionless parameter

$$\Lambda = -\frac{2\rho^2 \Omega (\Omega r^2)' c^4}{\sigma^2 H^4 r}; \quad \Lambda_{cr} \sim 1$$

Taylor vortices in a rotating conducting fluid in a magnetic field were investigated experimentally in [13]. It was found that the linear theory satisfactorily explains the experimental dependence of the critical rotational velocity on the magnetic field.

HELICAL INSTABILITY

Helical instability arises from the simultaneous action of sufficiently strong parallel, or nearly parallel, electric (E) and magnetic (H) fields on a non-uniform plasma [14].

Let us consider the propagation of a plane wave in such a plasma (all quantities in the wave vary as $e^{-i\omega t+ikx}$) and assume that

1. $\Omega_{e,i}\,\tau_{e,i} \ll 1$, where $\Omega = \frac{eH}{mc}$ and $1/\tau$ is the collision rate with neutral particles;
2. the wavelength is much smaller than the characteristic linear dimensions corresponding to a charge in the plasma parameters, $\frac{1}{k}\frac{\nabla_0 N}{N} \ll 1$;
3. the oscillations are low-frequency ($\omega\tau \ll 1$), potential oscillations ($E = -\nabla\Phi$), and the plasma is always quasi-neutral ($N_+ = N_-$).

The equations of motion

$$-\frac{T\nabla N}{N} - \frac{m\mathbf{U}}{\tau} + e\mathbf{E} + \frac{e}{c}\,\mathbf{U}\times\mathbf{H} = 0$$

then yield

$$U_\alpha = \left(\frac{D\nabla_\beta N}{N} + \mu\mathscr{E}_\beta\right)(\delta_{\alpha\beta} + \Omega\tau\varepsilon_{\alpha\beta\gamma}h_\gamma)$$

where $D = \frac{T\tau}{m}$ is the diffusion coefficient, $\mu = \frac{e\tau}{m}$ is the mobility, $\mathbf{h} = \mathbf{H}/H$ and $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric unit tensor. Substituting this expression in the continuity equation, $\frac{\partial H}{\partial t} + \nabla_\alpha N U_\alpha = 0$, we obtain an equation for the plasma charge density $N_\pm$:

$$\frac{\partial H}{\partial t} + \nabla_\alpha(-D\nabla_\beta N + \mu N\mathscr{E}_\beta)(\delta_{\alpha\beta} + \Omega\tau\varepsilon_{\alpha\beta\gamma}h_\gamma) = 0 \quad (3.42)$$

Linearising this equation, i.e. assuming that $N = N + n$, $\mathscr{E} = E - ik\varphi$, $\nabla = \nabla_0 + ik$, keeping those terms in (3.42) whose amplitude is proportional to the density n and potential φ, and considering that $\nabla_0 E = 0$, we obtain the following two relations between n and φ (here and below the subscript 0 in ∇_0 has been dropped):

$$-i\omega n + (D_\pm k_\alpha k_\beta n + \mu_\pm N k_\alpha k_\beta\varphi - \mu_\pm ik_\beta\varphi\nabla_\alpha N$$
$$+ \mu_\pm E_\beta ik_\alpha n)(\delta_{\alpha\beta} + \Omega_\pm\tau_\pm\varepsilon_{\alpha\beta\gamma}h_\gamma) = 0$$

Setting the determinant of this system of two algebraic equations for n and φ equal to zero, we obtain the dispersion relation connecting the frequency ω and wave vector $\mathbf{k}$:

$$A_+B_- - A_-B_+ = 0 \tag{3.43}$$

where

$$A = -\omega + \mu \mathbf{k}\cdot\mathbf{E} - i\,\frac{\mu T}{e}\,k^2 + \mu\Omega\tau\mathbf{k}\cdot(\mathbf{E}\times\mathbf{h})$$

$$B = \mu k^2 N - \mu i \mathbf{k}\cdot\nabla N - \mu\Omega\tau i \mathbf{k}\cdot(\nabla N \times \mathbf{h})$$

The imaginary part of the complex frequency $\omega + i\gamma$ can be obtained from (3.43):

$$\gamma = \frac{\mu_+|\mu_-|}{\mu_+ + |\mu_-|}\,(k^2N)^{-1}\Big(-\frac{2T}{|e|}\,k^4N - (\Omega_+\tau_+ + |\Omega_-\tau_-|)\,[(\mathbf{k}\cdot\mathbf{E})\mathbf{k}\cdot(\nabla N \times \mathbf{h}) - (\mathbf{k}\cdot\nabla N)\mathbf{k}\cdot(\mathbf{E}\times\mathbf{h})]\Big)$$

If, for example, the vectors $\mathbf{E} \parallel \mathbf{H}$ be along the z axis, whereas ∇N lies along the x axis, the waves will build up if

$$\frac{\Omega_+\tau_+ + |\Omega_-\tau_-|}{2}\cdot\frac{|e|E\lambda}{T}\cdot\frac{\lambda\nabla_x N}{N}\,\frac{k_y k_z}{k} > 1 \quad \left(\lambda \equiv \frac{1}{k}\right)$$

It follows from this that for given k_z, waves with only one sign of k_y will grow, i.e. the growing perturbations are in fact helical.

Experimental studies of helical instabilities show that the theory fully explains the properties observed. The theoretical dependence of the critical magnetic field on the electric field ($H_{cr} \sim E^{-1}$) has been verified, as has the dependence of the critical frequency (at which damping goes over to amplification) on E[15]. The distortion of the current during the onset of instability was investigated in [16] and it was shown that the resulting perturbation was helical

It was established experimentally in [17] that it is possible to stabilise a helical instability in a solid-state electron-hole plasma by introducing an external azimuthal magnetic field which increases towards the periphery of a cylindrical column of plasma. This field was produced by a current, flowing in external conductors parallel to the axis of the plasma column.

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Chapter 4

The Origin of Turbulence

The amplitude of small perturbations growing in an unstable plasma increases exponentially with time in such a way that the square of the amplitude, η, satisfies the differential equation

$$\frac{d\eta}{dt} = 2\gamma\eta \tag{4.1}$$

where γ is the growth rate in the linear theory. As the perturbation increases, its rate of growth decreases and (4.1) becomes invalid. If η is not large, the rate of its increase may be found by expanding the equation describing the plasma dynamics in terms of the perturbation amplitude. Assuming that the growth rate of the mean square amplitude is a regular function of the square of the amplitude, we have

$$d\eta/dt = 2\gamma_H\eta; \quad \gamma_H = \gamma + a\eta + b\eta^2 + \ldots, \tag{4.2}$$

This differs from (4.1) by the presence of the 'non-linear growth rate' γ_H, depending on the square of the amplitude

η instead of the corresponding quantity from the linear theory. Equation (4.2) may be used to describe a number of phenomena which take place in slightly supercritical unstable systems, i.e. when the growth rate is small.

There exist two turbulence modes, that is to say 'soft' and 'hard' (see Appendix A). The soft mode corresponds to $a < 0$ in (4.2). The square of the amplitude of steady-state motion in an unstable plasma is then given by

$$\eta = -\frac{\gamma}{a} + \ldots$$

and increases smoothly from zero as the system changes from stable ($\gamma < 0$) to unstable ($\gamma > 0$) state, and as the supercriticality increases. If the transition from the subcritical to the supercritical region is effected by changing a parameter X of the system, where X may be an electric or magnetic field, a temperature gradient, etc., the dependence on this parameter of any quantity Y, averaged over the pulsations, will have a discontinuity at the critical point $X = X_{cr}$.

In fact, the expansion of any quantity averaged over the amplitude of perturbations arising in unstable plasma is of the form

$$Y = Y_0 + \alpha\eta + \ldots \tag{4.3}$$

and, in as much as $\eta = 0$ for $X < X_{cr}$ and $\eta \backsim X - X_{cr}$ for $X > X_{cr}$, the derivative of Y with respect to X has a finite discontinuity (step) $\Delta(\partial Y/\partial X)$ at the point $X = X_{cr}$ (the function $Y(X)$ itself is, however, continuous).

If two parameters $X_{1,2}$ vary, the transition to unstable state occurs along some curve $\Phi(X_{1cr}, X_{2cr}) = 0$. Differentiating $Y_{1,2}(X_{1cr} + 0, X_{2cr} + 0) = Y_{1,2}(X_{1cr} - 0, X_{2cr} - 0)$ along the transition curve, we obtain

$$\begin{aligned} \Delta\frac{\partial Y_1}{\partial X_1} &= -\frac{dX_{2cr}}{dX_{1cr}}\Delta\frac{\partial Y_1}{\partial X_2} \\ \Delta\frac{\partial Y_2}{\partial X_1} &= -\frac{dX_{2cr}}{dX_{1cr}}\Delta\frac{\partial Y_2}{\partial X_2} \end{aligned} \tag{4.4}$$

Eliminating the quantity $\frac{dX_{2cr}}{dX_{1cr}}$ from (4.4), we find the relationship between the discontinuous changes in the derivatives of any two quantities averaged over the pulsations [1]:

$$\Delta \frac{\partial Y_1}{\partial X_1} \Delta \frac{\partial Y_2}{\partial X_2} = \Delta \frac{\partial Y_1}{\partial X_2} \Delta \frac{\partial Y_2}{\partial X_1} \tag{4.5}$$

In the 'hard' mode ($a > 0$), the turbulence arises as follows. A change in some plasma parameter X may result in a decrease in the growth rate ($\gamma \to 0$ as $X \to X_{cr}$) and the system may go over to the unstable state. For $\gamma = +0$, the amplitude of the perturbation abruptly reaches a finite value $\sqrt{\eta_1}$, determined by the vanishing of the non-linear growth rate;

$$a\eta_1 + b\eta_1^2 + \ldots = 0, \qquad \eta_1 = -\frac{a}{b} \tag{4.6}$$

Further increase in supercriticality results in a smooth increase in the amplitude from $\sqrt{\eta_1}$ to a larger value $\sqrt{\eta_0}$, determined by $\gamma_H(\eta_0) = 0$. If now the parameter X decreases and becomes lower than the critical value ($X < X_{cr}$), the motion in the plasma will not disappear. The amplitude $\sqrt{\eta}$ will fall abruptly to zero only for $X = X'_{cr} < X_{cr}$. It follows from (4.2) that the solution η_0 is stable if $(\partial\gamma_H/\partial\eta)_{\eta=\eta_0} < 0$, and unstable if $(\partial\gamma_H/\partial\eta)_{\eta=\eta_0} > 0$, so that the perturbations cut off for $(\partial\gamma_H/\partial\eta) = 0$, i.e. for $\eta = \eta_2 = -\frac{a}{2b}$. Thus in the hard mode of turbulence formation there is a hysteresis effect; the rise and collapse of pulsations in a plasma occur at different values of the external parameter.

The dependence on X of any observed quantity Y, averaged over the oscillations, in the case of hard turbulence excitation, exhibits discontinuities and hysteresis. The discontinuous changes Δ_1 and Δ_2 on the corresponding curve (excitation and cut-off of oscillations) are connected by the simple relation [2]

$$\Delta_1/\Delta_2 = 2 \tag{4.7}$$

derived from (4.3) by taking into account the equation $\eta_1 = 2\eta_2$.

The soft-mode discontinuities on the above curve have been repeatedly observed in experiments. The following may serve as examples:

1. the discontinuity in the heat flow at the onset of convection in a conducting fluid in a magnetic field [3];
2. the 'break' in the 'effective viscosity' curve during the formation of vortices in a conducting fluid in a

magnetic field between rotating coaxial cylinders [4];

3. the discontinuity in the hydrodynamic drag during the appearance of turbulence in magnetohydrodynamic flow [5];
4. the discontinuity in the dependence of current on voltage at the transition from subsonic to ultrasonic motion in solid-state plasma [6, 7].

The phenomenon of hysteresis (hard mode) in unstable plasma has been observed in the growth of helical (screw) perturbations in parallel electric and magnetic fields [8].

To find the form of motion being established in a plasma when $X > X_{cr}$, and to determine the wave vector, frequency and amplitude of the resulting almost periodic motion, it is necessary to consider in more detail the non-linear terms in the equations describing a given system.

Experiment shows that in many cases in which the supercriticality is small, travelling waves arise in a plasma with an amplitude which increases with increasing supercriticality. If the problem is essentially one-dimensional, e.g. in the case of the striations in the positive column of a glow discharge, all the quantities in the travelling wave are functions of one variable $\theta = \omega t - kx$. The form of the wave may then be found by solving standard differential equations [9].

The motion arising in a plasma under supercritical conditions has been investigated for magnetic convection [3], helical perturbations in the positive column of a glow discharge in a magnetic field [10] and for striations [11]. Results of these computations are in agreement with experimental data.

As the unstable plasma becomes more highly supercritical, higher harmonics are excited, and the motion gradually becomes highly turbulent. Theoretical studies of turbulent convection in plasma in a magnetic field are summarised and compared with experiment in [12].

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Chapter 5

Interaction of Plasmons with Resonance Particles

A characteristic property of plasma is the existence of a spectrum of collective oscillations, i.e. plasma waves or plasmons. The frequency and velocity of propagation of these waves is determined by the magnitude of the wave vector and the 'gross' parameters of the plasma, i.e. density, mean velocity spread, magnetic field and so on. This is a reflection of the fact that all the particles participate collectively in the plasma oscillations. The rate of damping or growth of these oscillations, on the other hand, depends on the detailed form of the particle velocity distribution in phase space, for example, the derivatives of the velocity distribution function. This shows the specific role of resonance particles which can amplify or damp the plasma waves as a result of energy transfer. Resonance particles fulfill the condition $\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{v} = n\omega_H$ where $n = 0, 1, 2 \ldots, \omega_{\mathbf{k}}$ and $\mathbf{k}$ are the frequency and wave vector of the oscillations, $\mathbf{v}$ is the particle velocity and $\omega_H = eH/mc$.

Quasi-linear theory, which will be considered below, describes the interaction between resonance particles and waves when the energy concentrated in the collective degrees

of freedom (plasma oscillations) is considerably smaller than the energy of random motion of all the particles, but at the same time is much greater than that of the thermal noise in the collective degrees of freedom.

The basis of the quasi-linear theory is the separation of the distribution function for resonance particles into a rapidly oscillating and slowly varying part, and the analysis of the influence of the root mean square of the oscillating part on the slowly varying one. This is similar to Van der Pohl's method in non-linear mechanics. It is found that the behaviour of the slowly varying part of the distribution function is described by a diffusion equation in phase space, and the rate of growth or decay of the rapid oscillations (plasma oscillations) is determined by the formulae of the linear theory in which the non-oscillatory part of the distribution function varies slowly with time.

In a homogeneous tenuous plasma in which collisions between particles are not very important, the stationary velocity distribution function is largely arbitrary. Quasi-linear theory indicates the existence of special states for unstable plasma due to the growth of perturbations. These states are characterised by a distribution function f which is constant in certain regions in phase space (it exhibits a number of 'plateaus'). The noise level in the corresponding regions of wave-number space is much higher than the thermal level.

EQUATIONS OF QUASI-LINEAR THEORY

We shall now derive the basic equations of quasi-linear theory for a fully ionised tenuous plasma. To be specific, let us consider longitudinal Langmuir oscillations of electrons. We shall start with the equation including the self-consistent field

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{eE}{m}\frac{\partial f}{\partial v} \tag{5.1}$$

and the equation

$$4\pi en\int vf dv = -\frac{\partial E}{\partial t} \tag{5.2}$$

where f is the electron distribution function and n is the

equilibrium plasma density (in equilibrium $\int f_0 dv = 1$). We know from the linear theory that solutions of (5.1) and (5.2) which are harmonic in time describe Langmuir oscillations. The damping rate of these oscillations in equilibrium plasma is small compared with their frequency if their wavelength is considerably greater than the Debye radius. If we transform to the space Fourier components

$$f = f_0 + \sum_k G_k e^{ikx}$$

$$E = \sum_k E_k e^{ikx}$$

in the linearised equations (5.1) and (5.2) and neglect the term $v \frac{\partial f}{\partial x}$ for long-wavelengths, we obtain

$$\dot{G}_k = -\frac{e}{m} E_k \frac{\partial f_0}{\partial v}$$

$$\dot{E}_k = -4\pi en \int v G_k dv$$

from which it follows that the long-wave spectral component of the field satisfies the oscillator equation

$$\ddot{E}_k = -\omega_p^2 E_k \tag{5.3}$$

with the Langmuir frequency $\omega_p = \sqrt{4\pi ne^2/m}$. All the plasma particles participate in these oscillations and the kinetic energy of all of them is equal to the electrostatic energy (virial theorem)

$$\frac{1}{2} nm \left|\int v G_k dv\right|^2 = \frac{E_k^2}{8\pi}$$

so that the total energy of the oscillations is equal to $E_k^2/4\pi$. An oscillation of wave number k is not damped if there are no plasma particles with velocity $v \gtrsim \omega_p/k$ (the term $v\partial f/\partial x \backsim kvG_k$ can then be neglected in comparison with $\partial f/\partial t \backsim \omega_p G_k$, as was in fact done in the derivation of (5.3)).

If the plasma contains electrons with velocities equal to the phase velocity ω_p/k of any Langmuir wave, it becomes possible for energy to be exchanged between the waves and these 'resonance' electrons. We shall assume that the

number of resonance particles is small, and will neglect changes in the dispersive properties of the plasma (oscillation frequency, phase and group velocities, but not damping) which are due to the resonance particles, so that, even when resonance electrons are present, we shall consider the frequency of the Langmuir oscillations to be equal to ω_p.

Interactions between plasma oscillations and resonance particles lead to two effects, a change in the average energy of the Langmuir oscillations $E_k^2/4\pi$ and a simultaneous change in the distribution of the resonance particles in velocity space. To derive equations describing these processes we proceed as follows. The distribution function F for resonance particles is written in the form of a sum of a rapidly oscillating part $\sum_k F_k e^{ikx}$ and a slowly varying function $\bar{F}$. The electric fields $E = \sum_k E_k e^{ikx}$ will then be products of functions oscillating rapidly in time and space and of slowly varying amplitudes (it is assumed that the average field is zero.) The averages of the oscillating part of the distribution function and of the electric field, evaluated over time intervals considerably greater than the period of the plasma oscillations, are also zero, i.e.

$$\langle F_k \rangle = \langle E \rangle = 0$$

so that $\bar{F}$ is the average total distribution function for resonance electrons. Averaging the kinetic equation for resonance electrons

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} = -\frac{eE}{m}\frac{\partial F}{\partial v} \tag{5.4}$$

over all space we obtain

$$\frac{\partial \bar{F}}{\partial t} = -\overline{\frac{eE}{m}\frac{\partial F}{\partial v}} = -\frac{\partial}{\partial v}\frac{e}{m}\sum_k E_k^+ F_k \tag{5.5}$$

Further, subtracting (5.5) from (5.4) and neglecting the difference $\overline{E\frac{\partial F}{\partial v}} - E\frac{\partial F}{\partial v}$, (this difference yields terms in (5.6) which are non-linear in E and represent interactions between plasma waves), we obtain

$$\dot{F}_k + ikvF_k = -\frac{eE_k}{m}\frac{\partial \bar{F}}{\partial v} \tag{5.6}$$

From the kinetic equation for non-resonance particles we have

$$\dot{G}_k = -\frac{eE_k}{m}\frac{\partial f_0}{\partial v}$$

and consequently the time derivative of the current density due to all the plasma particles except those in resonance is given by

$$ne\int vG_k dv = \frac{ne^2}{m}E_k$$

Taking this into account, we obtain from the equation for the total current (5.2),

$$\ddot{E}_k + \omega_p^2 E_k = 4\pi en\int v\dot{F}_k dv \tag{5.7}$$

Integrating (5.6) we have

$$F_k(t) = F_k(0)e^{-ikvt} - \frac{e}{m}\int_0^t dt' E_k(t')\frac{\partial \overline{F}(t')}{\partial v}e^{-ikv(t-t')} \tag{5.8}$$

and substituting this expression into (5.7), multiplying both sides of the resulting equality by $\dot{E}_k^+$ and combining it with its complex conjugate expression, we obtain

$$\dot{E}_k^+(\ddot{E}_k + \omega_p^2 E_k) + \text{c.c.} = \frac{d}{dt}(|\dot{E}_k|^2 + \omega_p^2|E_k^2|)$$

$$= 4\pi en\dot{E}_k^+\int vdv\left\{-ikvF_k(0)e^{-ikvt} - \frac{eE_k(t)}{m}\frac{\partial\overline{F}(t)}{\partial v}\right.$$

$$\left. + \frac{e}{m}ikv\int_0^t dt' E_k(t')\frac{\partial F(t')}{\partial v}e^{-ikv(t-t')}\right\} + \text{c.c.}$$

Substituting $E_k(t) = \sqrt{\varepsilon_k(t)}\,e^{-i\omega_p t}$ into this, and taking the slowly varying functions ε_k and $\partial\overline{F}/\partial v$ outside the integral with respect to t', we obtain an equation for the square of the wave amplitude:

$$\frac{d}{dt}2\omega_p^2\varepsilon_k = -4\pi nei\omega_p\sqrt{\varepsilon_k}\int vdv\left\{-ikvF_k(0)e^{-ikv}\right.$$

$$\left. + ikv\frac{e}{m}\sqrt{\varepsilon_k}\frac{\partial\overline{F}}{\partial v}\int_0^t e^{i(\omega_p - kv)(t-t')}dt'\right\} + \text{c.c.}$$

As $t\to\infty$, the first term in curly brackets disappears,

whereas the second yields

$$\frac{d\varepsilon_k}{dt} = \varepsilon_k \pi \omega_p^2 \int v \frac{\partial \overline{F}}{\partial v} \delta(\omega_p - kv)\, dv$$

in accordance with the formula

$$\lim_{t\to\infty} \left(\int_0^t e^{i\alpha(t-t')} dt' + \text{c.c.} \right) = 2\pi\delta(\alpha)$$

i.e. the energy growth rate in a given harmonic is

$$\frac{d\varepsilon_k}{dt} = 2\gamma_k \varepsilon_k \tag{5.9}$$

where

$$\gamma_k = \frac{\pi}{2} \frac{\omega_p^3}{k^2} \int k \frac{\partial F}{\partial v} \delta(\omega_p - kv)\, dv \tag{5.10}$$

We thus see that in the quasi-linear theory the rate of increase (or decrease) in the energy of a particular Fourier harmonic is in fact determined by the same formula as in the linear theory, except that the average function $\overline{F}$ replaces the 'unperturbed' distribution function.

The second equation of the quasi-linear theory is obtained by substituting (5.8) into (5.5) and combining the result with its complex conjugate:

$$\frac{\partial \overline{F}}{\partial t} = -\frac{1}{2} \frac{\partial}{\partial v} \frac{e}{m} \sum_k E_k^+ \left\{ F_k(0)\, e^{-ikvt} \right.$$
$$\left. - \frac{e}{m} \int_0^t dt' E_k(t') \frac{\partial \overline{E}(t)}{\partial v} e^{-ikv(t-t')} \right\} + \text{c.c.}$$

Similarly, replacing $E_k(t)$ by $\sqrt{\varepsilon_k(t)}\, e^{-i\omega_p t}$, we obtain the following equation for the average resonance-particle distribution function $\overline{F}$:

$$\frac{\partial \overline{F}}{\partial t} = \frac{\partial}{\partial v} \frac{e^2}{m^2} \sum_k \varepsilon_k \pi \delta(\omega_p - kv) \frac{\partial \overline{F}}{\partial v}$$

The second equation is thus of the form [1 - 3]

$$\frac{\partial \overline{F}}{\partial t} = \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial \overline{F}}{\partial v_\beta} \tag{5.11}$$

where
$$D_{\alpha\beta} = \pi \frac{e^2}{m^2} \sum_k \varepsilon_k \delta(\omega_p - kv) \frac{k_\alpha k_\beta}{k^2} \tag{5.12}$$

Equations (5.9) to (5.11) form a closed system of equations for the spectral density ε_k and the average distribution function $\bar{F}$ (henceforth we shall omit the bar over the distribution function). Equation (5.11) has the form of a diffusion equation. According to (5.12), the diffusion coefficient D is proportional to the energy density of the plasma oscillations, which in turn depends on the distribution function.

It is readily verified that the system of quasi-linear equations (5.9) to (5.12), describing the interaction of resonance particles with plasma oscillations, has an energy integral. In fact, let us consider the expression for the time derivative of the total energy Q of a system of resonance particles and oscillations. The quantity Q consists of the kinetic energy of the resonance electrons, the energy of the electrostatic field of the plasma oscillations, $\sum_k \frac{\varepsilon_k}{8\pi}$, and the kinetic energy of all the plasma electrons which take part in these oscillations. This last energy is equal to the electrostatic energy (by the virial theorem). Thus

$$\frac{dQ}{dt} = \frac{d}{dt}\left(n \int \frac{mv^2}{2} F\, dv + \sum_k \frac{\varepsilon_k}{4\pi}\right)$$

Substituting for $\partial F/\partial t$ from (5.11) and for $d\varepsilon/dt$ from (5.9) into this expression, and integrating by parts, we obtain

$$\frac{dQ}{dt} = 0$$

i.e. the total energy of a system of plasmons and particles is conserved.

To clarify the physical significance of the quasi-linear theory, and to generalise Equations (5.9) to (5.12), let us consider a plasma with strongly excited collective degrees of freedom as a combination of two gases: a gas of particles in a non-degenerate state (fermions) and a gas of plasmons (bosons).

Let us consider the balance equation for the number of particles and waves (plasmons) in phase space, assuming that the system is homogenous. The basic process which must be taken into account is induced radiation or absorption of a plasmon with wave vector $\mathbf{q}$ by a particle with wave vector $\mathbf{k}$. The first process is Cherenkov emission of a

plasmon by an electron travelling in a plasma with velocity v, greater than the phase velocity of the plasma wave ω_q/q:

$$v = \frac{\omega_q}{q}\cos\theta$$

where θ is the angle between $\mathbf{q}$ and $\mathbf{v}$; the inverse process is Cherenkov absorption of plasmons by particles.

The matrix elements of these induced processes are proportional to $\sqrt{N_\mathbf{q}}$, and consequently the probability W of both processes is the same, and given by

$$W(\mathbf{k}, \mathbf{q}) = N_\mathbf{q} w_{\mathbf{k},\mathbf{k}+\mathbf{q}} \delta(\varepsilon_i - \varepsilon_f), \quad w_{\mathbf{k},\mathbf{k}+\mathbf{q}} = w_{\mathbf{k}+\mathbf{q},\mathbf{k}}$$

where $\varepsilon_{i,f}$ is the energy of the initial and final states repectively.

As a result of emission or absorption of waves, the particle momentum will change and move to a different point in phase space. The change in the number of particles at a point in phase space consists of 'exit' terms resulting from the absorption and emission of plasmons

$$-\sum_\mathbf{q} F_\mathbf{k} N_\mathbf{q} w_{\mathbf{k},\mathbf{k}+\mathbf{q}} \delta(\varepsilon_\mathbf{k} + \omega_\mathbf{q} - \varepsilon_{\mathbf{k}+\mathbf{q}})$$

$$-\sum_\mathbf{q} F_\mathbf{k} N_\mathbf{q} w_{\mathbf{k},\mathbf{k}-\mathbf{q}} \delta(\varepsilon_\mathbf{k} - \omega_\mathbf{q} - \varepsilon_{\mathbf{k}-\mathbf{q}})$$

and of the analogous 'entrance' terms due to absorption and emission of plasmons

$$+\sum_\mathbf{q} F_{\mathbf{k}-\mathbf{q}} N_\mathbf{q} w_{\mathbf{k}-\mathbf{q},\mathbf{k}} \delta(\varepsilon_{\mathbf{k}-\mathbf{q}} + \omega_\mathbf{q} - \varepsilon_\mathbf{k})$$

$$+\sum_\mathbf{q} F_{\mathbf{k}+\mathbf{q}} N_\mathbf{q} w_{\mathbf{k}+\mathbf{q},\mathbf{k}} \delta(\varepsilon_{\mathbf{k}+\mathbf{q}} - \omega_\mathbf{q} - \varepsilon_\mathbf{k})$$

Here $F_\mathbf{k}$ is the distribution function for particles in phase space, $\varepsilon_\mathbf{k}$ is the kinetic energy of a particle with wave vector $\mathbf{k}$ and $\omega_\mathbf{q}$ is the energy of the wave $\mathbf{q}$.

Summing the contributions of the various processes, we obtain the following equation for the particle distribution function $F_\mathbf{k}$:

$$\partial F_\mathbf{k}/\partial t = \sum_\mathbf{q} N_\mathbf{q}(\Psi_{\mathbf{k}+\mathbf{q},\mathbf{q}} - \Psi_{\mathbf{k},\mathbf{q}}) \tag{5.13}$$

where

$$\Psi_{\mathbf{k},\mathbf{q}} = (F_\mathbf{k} - F_{\mathbf{k}-\mathbf{q}}) w_{\mathbf{k},\mathbf{k}-\mathbf{q}} \delta(\varepsilon_\mathbf{k} - \varepsilon_{\mathbf{k}-\mathbf{q}} - \omega_\mathbf{q})$$

The equation for the wave distribution $N_{\mathbf{q}}$ may be found in a similar way, Changes in $N_{\mathbf{q}}$ take place as a result of the same processes of emission and absorption of plasmons by particles as in the specially uniform case already discussed:

$$\partial N_{\mathbf{q}}/\partial t = N_{\mathbf{q}} \sum_{\mathbf{k}} \Psi_{\mathbf{k}+\mathbf{q},\mathbf{q}} \tag{5.14}$$

Equations (5.13) and (5.14), which may be derived (see Appendix B) from the equations for plasma density matrices. can be used to obtain, for example, Equations (5.9) to (5.11) describing the interaction of Langmuir plasmons with electrons in tenuous plasma. For this it is sufficient to take into account the fact that the relative change in the momentum of a particle during the creation (or absorption) of a wave in tenuous plasma is very small ($q/k \to 0$), and to use the following formulae for the probability w and the number of quanta of Langmuir oscillations $N_{\mathbf{q}}$:

$$w_{\mathbf{k},\mathbf{k}-\mathbf{q}} = \frac{4\pi^2 e^2 \omega_p^2}{\mathbf{q}^2}, \quad N_{\mathbf{q}} = \frac{|E_{\mathbf{q}}|^2}{4\pi\omega_p}$$

In this case (5.13) becomes identical with (5.11), and (5.14) goes over to the formula for the growth rate (5.9).

Equations (5.13) and (5.14) describe the interaction between plasmons and particles in weakly turbulent plasma.

RELAXATION OF PLASMA OSCILLATIONS

Let us apply the quasi-linear theory to the problem of damping of plasma oscillations. Linear theory predicts an exponential damping with a time constant of the order of $1/\gamma$. The damping rate γ is determined in the linear theory by the thermodynamic equilibrium (Maxwellian) distribution function, since it is assumed that at the moment of creation of oscillations the plasma is in a state of thermodynamic equilibrium.

If, however, at the initial moment the energy of the plasma oscillations which are created is considerably greater than the energy of thermal noise in the equilibrium plasma, the process of damping will proceed quite differently. As long as the energy density ε of the waves is much greater than the thermal noise energy density nT/N_D,

collisions between plasma particles do not play an appreciable role, whereas wave diffusion and the levelling-off of the distribution function take place in the region of velocity space in which there are particles with the resonance velocities. As a result, the particles acquire higher velocities and simultaneously with the damping of plasma oscillations, the kinetic energy of the particles increases, although the total energy of the system of waves and particles is conserved. This process of quasi-linear absorption is completed when γ vanishes. At the same time the energy of the plasma oscillations remains, generally speaking, finite and much greater than the thermal noise level. Thereafter the oscillations are no longer damped, since $\gamma = 0$, and the distribution function does not change. In reality, collisions between the particles give rise to slow diffusion in velocity space, leading finally to the Maxwellian equilibrium distribution and to the damping of the oscillations to the level of thermal noise. This second phase occurs over a far longer time than the first.

Let us consider the case of electron Langmuir waves. We shall assume that at time $t = 0$ the plasma is in thermodynamic equilibrium (Maxwellian velocity distribution) and is homogeneous in all space, and that plasma oscillations are created with a range of wave vectors **k** and an energy density $\varepsilon_k(0)$ considerably greater than the thermal noise energy. We shall consider that all the wave vectors **k** are parallel to one another, i.e. we shall be dealing with a one-dimensional spectrum of oscillations. An analytical solution of the problem is then possible. For a one-dimensional spectrum and sufficiently long waves, the velocity of the resonance particles is uniquely related to the wave vector by the simple formula

$$v = \omega_0/k$$

where ω_0 is the Langmuir frequency. The coefficient of wave diffusion is, in this case,

$$D(v) = \frac{e^2}{2m^2} \frac{|E_k^2|}{v}$$

and the damping rate is defined by the expression

$$\gamma = \frac{\pi}{2} \frac{\omega^3}{k^2} \frac{\partial f}{\partial v}$$

where f is the distribution function for plasma electrons

averaged over the velocity component in the direction of the wave vector k (it is normalised so that $\int f dv = 1$).

The equations of the quasi-linear theory thus take the form

$$\partial \varepsilon / \partial t = A \varepsilon \, \partial f / \partial v \tag{5.15}$$

$$\partial f / \partial t = \frac{\partial}{\partial v} \left(B \varepsilon \frac{\partial f}{\partial v} \right) \tag{5.16}$$

where $\varepsilon = E_k^2 / 8\pi$, f is a function of time t and velocity $v = \omega_0 / k$, and the coefficients A and B depend on the velocity, but are independent of time:

$$A = \pi \omega_0^2 v^2; \quad B = \omega_0^2 / nmv \tag{5.17}$$

The initial conditions for (5.15) and (5.16) are: at $t = 0$, $\varepsilon = \varepsilon_0(v)$, $f = f_M(v)$, the spectral density $\varepsilon_0(v)$ differs from zero in the finite range of velocity $v_1 < v < v_2$ and f_M is the Maxwellian distribution function.

Under the action of wave diffusion, in the region $v_1 < v < v_2$ the negative derivative of the distribution function increases, i.e. the distribution function becomes more flat. This is accompanied by damping of the waves and a decrease in the diffusion coefficient. If the initial noise density $\varepsilon_0(v)$ is sufficiently large, $\partial f / \partial v$ vanishes as a result of the relaxation process, but the noise density $\varepsilon_\infty(v)$ remains finite. The system goes over to a state in which $\partial f / \partial v = 0$ in the range $v_1 < v < v_2$ and $f = f_M$ outside this range. The diffusion coefficient D (and the energy density of plasma oscillations) will differ from zero in the region $v_1 < v < v_2$ and will remain equal to zero outside this region. In accordance with the quasi-linear theory, a state with a plateau in the distribution function should be stationary since Equations (5.15) and (5.16) are then satisfied over all velocity space. In reality, collision processes, which were not taken into account in (5.15) and (5.16), will lead, as we have seen, to a slow diffusion of particles in velocity space and eventually to the establishment of thermodynamic equilibrium. Thus, the above distribution with a plateau is quasi-stationary. If, however, we do not consider these slow processes, we may speak of a stationary final state.

The quasi-linear equations (5.15) and (5.16) allow us to relate the energy density spectrum of plasma oscillations $\varepsilon_\infty(v)$ in the stationary state to the initial spectral density $\varepsilon_0(v)$. In fact, substituting $\varepsilon \partial f / \partial v = A^{-1} \partial \varepsilon / \partial t$ from (5.15) into

(5.16), we can verify that the quantity $f - \frac{\partial}{\partial v} BA^{-1}\varepsilon$ is conserved in the relaxation process:

$$\frac{\partial}{\partial t}\left(f - \frac{\partial}{\partial v} BA^{-1}\varepsilon\right) = 0 \tag{5.18}$$

In particular, in the final state $(t \to \infty)$

$$f_\infty - \frac{\partial}{\partial v} BA^{-1}\varepsilon_\infty = f_M - \frac{\partial}{\partial v} BA^{-1}\varepsilon_0$$

so that

$$\varepsilon_\infty(v) = \varepsilon_0(v) - AB^{-1}\int_{v_1}^{v} (f_M - f_\infty)\, dv \tag{5.19}$$

Since the height of the plateau f_∞ is a known constant quantity (it is determined by the conservation of the total number of resonance particles)

$$\int_{v_1}^{v_2} (f_M - f_\infty)\, dv = 0, \quad \text{i.e.} \quad f_\infty = (v_2 - v_1)^{-1} \int_{v_1}^{v_2} f_M\, dv$$

the relationship given by (5.19) determines the energy density spectrum for oscillations in the final stationary state. The law of conservation of resonance particles follows from that for the conservation of the total number of particles; since the distribution function $f(v)$ is the same for $v < v_2$ and $v > v_1$, the total number of resonance particles $(v_1 < v < v_2)$ must be conserved.

The reduction in the energy of the oscillations as a result of the process of quasi-linear relaxation is compensated by the increase in the kinetic energy of the particles which enter the region of higher velocities through diffusion in phase space. It follows from (5.19) that

$$\int_{\omega/v_2}^{\omega/v_1} \{2\varepsilon_0(v) - 2\varepsilon_\infty(v)\} \frac{dk}{2\pi} = \int_{v_1}^{v_2} dv'\, nmv' \int_{v_1}^{v_1} (f_M - f_\infty)\, dv$$

Integrating the right-hand side of this with respect to time, we obtain the law of conservation of energy

$$\int (2\varepsilon_0 - 2\varepsilon_\infty) \frac{dk}{2\pi} = \int_{v_1}^{v_2} n \frac{mv^2}{2} (f_M - f_\infty)$$

It may happen that the initial energy of the waves is insufficient for the establishment of a plateau on the electron distribution (Equation (5.19) then leads to a meaningless negative expression for ε_∞). A stationary state will not then be reached, and the system will arrive at a state of thermodynamic equilibrium in a time of the order of the time between binary particle collisions.

So far we have considered a tenuous plasma without taking particle collisions into account. These lead to a gradual disappearance of the plateau on the distribution function and to a transition of the system to a state of thermodynamic equilibrium. The quasi-linear equations for the distribution function with allowance for diffusion due to the emission and absorption of waves and to binary collisions, is of the form [2]

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + \frac{\partial}{\partial v} \nu \left(vf + \frac{T}{m} \frac{\partial f}{\partial v} \right) \tag{5.20}$$

where ν is the collision rate.

Integrating (5.20) with respect to time, and recalling that $D\partial f/\partial v = A^{-1}\partial D/\partial t$, where A is given by (5.17), we obtain

$$\left(f - \frac{\partial}{\partial v} A^{-1} D - \frac{\partial}{\partial v} \nu \frac{T}{m} A^{-1} \ln D \right) \Big|_0^t = \frac{\partial}{\partial v} \int_0^t dt\, \nu v f \tag{5.21}$$

If the change in the distribution function $f_0 - f_M$ is considerably smaller than $\partial D_0 A^{-1}/\partial v$, we may neglect the first term on the left-hand side. Integrating (5.21) with respect to v from $-\infty$ to v, using the fact that in the damping process the distribution function changes insignificantly (which of course cannot be said of its derivative df/dv), and substituting for f under the integral sign on the right-hand side of (5.21) the thermodynamic equilibrium function $f_M = (2\pi T/m)^{-1/2} \exp -mv^2/2T$, we obtain the following transcendental equation, defining the time dependence of the diffusion coefficient D (or the wave energy $\varepsilon = B^{-1}D$)

$$\left(-\frac{m}{T\nu} D - \ln D \right) \Big|_0^t = -A \frac{\partial f_M}{\partial v} t \tag{5.22}$$

It is clear from this equation that during the initial phase the noise is damped according to the linear law $D_0 - D \backsim t$, and exponential damping $\ln \frac{D_0}{D} \backsim t$ arises only during the last stage of the process, when the noise level becomes sufficiently small.

It follows that the effect of waves on the particles, which is accounted for in the quasi-linear theory, leads to a sharp reduction in absorption; resonance particles are redistributed, and a plateau is formed on the distribution function. Collisions gradually flatten the region of the plateau and a stationary state is established, in which

$$\frac{\partial f}{\partial v} = -\frac{\nu v f}{\nu \frac{T}{m} + D}$$

Since the result of this is a change in the derivative of the distribution function rather than in the function itself, it follows that

$$\frac{\partial f}{\partial v} = \frac{1}{1 + D \frac{m}{T\nu}} \frac{\partial f_M}{\partial v} \tag{5.23}$$

From (5.15) and (5.23) we find

$$2\gamma = \frac{1}{D} \frac{\partial D}{\partial t} = \frac{1}{1 + D \frac{m}{T\nu}} A \frac{\partial f_M}{\partial v} .$$

Integrating this ordinary differential equation for D (the velocity v enters as a parameter) we arrive again at (5.22).

The factor by which the damping rate in the linear theory is reduced because of the distortion of the distribution function may be written in the form

$$\frac{\gamma}{\gamma_0} = \frac{1}{1 + D \frac{m}{T\nu}} = \frac{1}{1 + \frac{e^2}{mT\nu k} \frac{E_k^2 \Delta k}{\Delta v}} \quad (5.24$$

and, consequently, in the case of 'monochromatic' waves (where $\Delta v \approx \sqrt{e\varphi_0/m}$, $E_k^2 \Delta k = E^2$ and φ_0 is the amplitude of the wave potential), it is equal to

$$\frac{\gamma}{\gamma_0}=\frac{1}{1+A\dfrac{(e\varphi_0)^{3/2}}{\sqrt{m}\,T\nu\lambda}} \tag{5.25}$$

where $A \sim 1$.

Let us now consider the problem of Langmuir waves crossing a plasma layer. The linear theory of small oscillations in tenuous plasma predicts collisionless damping of waves propagating in plasma. The onset of such damping is indicated by a decrease in the amplitude of longitudinal Langmuir waves excited on the plasma boundary by an external electric field with frequency $\omega > \omega_0$ and propagating into the plasma at right angles to the boundary. For long-wavelength oscillations (the only ones we shall consider), the decrease in the wave amplitude in the plasma is given by

$$\varepsilon_k^{-1}\frac{\partial\varepsilon_k}{\partial x}=\frac{\pi}{3}\frac{\omega_0^4}{k^3}\frac{m}{T}\frac{\partial f}{\partial v} \tag{5.26}$$

where $k=\omega/v$ is the wave vector, $\omega^2=\omega_0{}^2+3\frac{T}{m}k^2$ and f is the distribution function for electrons over the velocity component parallel to the direction of propagation of the wave and perpendicular to the boundary. This expression (valid only at distances of several wavelengths from the plasma boundary) follows from (5.9), if we take into account the fact that in the case under consideration

$$f=f_M=(2\pi T/m)^{-1/2}\exp-(mv^2/2T)$$

Thus, the linear theory, in which the energy of a wave packet is considered to be infinitely small, leads to an exponential damping of the energy of a wave packet with distance. The damping rate is determined by (5.26) with

$$\frac{\partial\varepsilon_k}{\partial t}\rightarrow\frac{\partial\omega_k}{\partial k}\frac{\partial\varepsilon_k}{\partial x}=3\,\frac{k}{\omega}\frac{T}{m}\frac{\partial\varepsilon_k}{\partial x}$$

In reality, the wave energy is finite, and the corresponding wave diffusion levels off the distribution function for resonance particles and reduces the damping. If we disregard collisions and take into account the fact that the parameter $N_D\,\varepsilon/nT$ (ε is the wave energy density) is much greater than unity, we find from the quasi-linear equations that at a certain distance from the boundary the waves will form a plateau on the distribution function:

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} = 0, \quad D \neq 0$$

and will propagate further without damping:

$$f(v, x) = \text{const}, \quad \frac{\partial \varepsilon_k}{\partial x} = 0$$

To obtain finite absorption, we must therefore take account of collisional terms in the equation for the particle distribution function

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + \frac{\partial}{\partial v} \nu \left(v f + \frac{T}{m} \frac{\partial f}{\partial v} \right) \qquad (5.27)$$

Equations (5.26) and (5.27) (with initial conditions $\varepsilon_k = \varepsilon_0(v)$, $f = f_0(v)$ at $x = 0$) determine the variation in the wave energy-density spectrum and the distribution function with distance. At first the energy of a wave packet falls linearly with distance from the boundary

$$\frac{\varepsilon_k(k)}{\varepsilon_k(0)} = 1 - \frac{x}{L}$$

(the characteristic length for damping, L, is directly proportional to the energy of the waves at the boundary; its order of magnitude is $L \approx k^{-1} \varepsilon N_D / nT$), and then, when the energy becomes sufficiently small, falls exponentially in accordance with (5.26).

The quantity L may be much greater than that damping length L_{lin} which is given by the linear theory. For wavelengths of the order of the Debye radius, $L/L_{\text{lin}} \approx \varepsilon N_D / nT$.

For waves propagating in aniso-tropic plasma, the formula for the quasi-linear damping rate becomes more complicated, but its structure remains the same as in (5.24) and (5.25):

$$\frac{\gamma}{\gamma_0} = \frac{1}{1 + \frac{\nu'}{\nu}}$$

Here ν is the collision rate for electrons and ν' is the reciprocal of the time for quasi-linear plateau formation.

GROWTH OF PERTURBATIONS IN AN UNSTABLE PLASMA

Let us consider the growth of perturbations in unstable plasma in terms of quasi-linear theory. We shall study the dynamical properties of a plasma, unstable with respect to the growth of electron Langmuir oscillations. To simplify the problem we will limit ourselves to the case where the wave vectors of the oscillations growing in the unstable plasma are all parallel, and the wave spectrum is one-dimensional. This occurs when the plasma has a preferred direction (external magnetic field, axis of the tube filled with plasma and so on) and when the growth rate of oscillations whose wave vector is parallel to this direction is a maximum (see Appendix D). We shall assume that the distribution function $f_0(v)$ for the plasma electrons is such that at the initial instant $\partial f/\partial v$ is positive in a small velocity interval (when the average velocity in this interval is considerably higher than the average thermal velocity of the plasma electrons). The plasma state is then unstable and the spectral energy density ε_k in the corresponding interval of wave number space $k = \omega_0/v$ begins to grow in accordance with (5.15). The growth of oscillations leads to an increase in the wave diffusion coefficient for resonance particles, the distribution function flattens off and there is a broadening of the region of instability.

The growth of oscillations and the diffusion of resonance electrons occurs until a plateau is formed on the distribution function, i.e. a region for which $\partial f/\partial v = 0$. The growth of the oscillations then ceases and a steady state is established. The magnitude of the electron distribution function in the region of the plateau in the final state, f_∞, may be found from the conservation of the total number of resonance particles diffusing in phase space to the region of lower velocities while the steady state is being established:

$$\int_{v_1}^{v_2} f(0, v)\, dv = (v_2 - v_1) f_\infty \tag{5.28}$$

The velocities $v_{1,2}$ determine the end-points of the plateau and can be found at the same time as f_∞ by simultaneous solution of (5.28) and the equations

$$f_0(v_1) = f_0(v_2) = f_\infty \tag{5.29}$$

This means that the areas under the curves $f_0(v)$ and f_∞ must be equal. Outside the plateau, the distribution function retains its initial values.

The noise spectral density in the final state is connected (as in the quasi-linear relaxation of Langmuir waves) with the initial spectral density ε_0 and with the change in the distribution function $f_\infty - f_0$ by

$$\varepsilon_\infty(v) - \varepsilon_0(v) = -AB^{-1}\int_{v_1}^{v}(f_0 - f_\infty)\,dv$$

where the functions A and B are defined by (5.17). If the noise level in the system at the initial time is thermal, we can neglect $\varepsilon_0(v)$, so that the spectrum of Langmuir oscillations in the final state will be determined only by the initial distribution of electrons near the region of the growth of f_0 [3]:

$$\varepsilon_\infty(v) = AB^{-1}\int_{v_1}^{v}(f_\infty - f_0)\,dv \tag{5.30}$$

The order of magnitude of the energy density of the oscillations established after the termination of the diffusion process is given by

$$\frac{E_\infty^2}{8\pi} \approx \delta n\,(mv_2^2 - mv_1^2)$$

where $v_{1,2}$ are the limits of the plateau and δn is the density of the group of electrons which assume the lower energies in velocity space.

The duration τ of the process of excitation of oscillations, and of relaxation of the electron distribution leading to the establishment of a plateau, may be evaluated in the same way as the diffusion time in velocity space by using the expression for the diffusion coefficient D_∞ in the final state:

$$\tau \approx \frac{(v_2 - v_1)^2}{D_\infty} \approx \frac{1}{\omega_0}\left(\frac{v_2 - v_1}{v}\right)^2 \frac{n}{\delta n}$$

where v is the average velocity on the plateau.

As a result of the growth of instability, the kinetic energy of the resonance particles is transformed into the energy of the electric field of Langmuir oscillations and the kinetic

energy of all the plasma electrons which take part in these collective oscillations. The total energy of the plasma is, however, conserved.

INTERACTION OF A BEAM WITH A PLASMA

A system consisting of a plasma and a stream of charged particles (beam) flowing through it can become unstable under certain conditions. Much experimental and theoretical work has been devoted to the study of this so-called electrostatic instability. Linear theory (Chapter 3) predicts that this instability has a somewhat different character in two limiting cases. When the beam is dense, almost mono-energetic and has a large velocity relative to the plasma, growing oscillations arise in the plasma with frequency and growth rate determined by the parameters of the whole system. If, however, the relative velocity and the density of the beam are not very large, and the spread of velocities in it is not too small, the frequency of the oscillations is equal to the Langmuir frequency and only the growth rate is determined by the properties of the whole system; it is proportional to the time derivative of the resultant distribution function for electrons in the plasma and beam (at the point $v = \omega/k$). Quasi-linear theory is useful only in the second case.

In the analysis of beam-plasma interactions we will limit ourselves, as before, to one-dimentional electron Langmuir waves. Let a beam move through a plasma in the positive direction of the x axis. The distribution functions for electrons of the plasma and the beam and the spectral density of the noise are prescribed at the point $x = 0$. The waves begin to grow when $\partial f/\partial v > 0$ for particles in resonance with plasma waves ($v = \omega/k$). At the same time, the beam and plasma electrons diffuse in velocity space, flattening the distribution function in the region where the wave diffusion coefficient differs from zero and leading to a decrease in the growth rate. As the beam passes through the plasma, the derivative of the distribution function with respect to velocity decreases, and the wave energy increases. As $x \to \infty$, the system reaches a steady state in which there is a plateau on the distribution function for electrons of the plasma-beam system, and undamped plasma oscillations arise in the corresponding wave-vector space. Since the

energy density of these oscillations is greater than at the entrance to the system (at $x = 0$), the kinetic energy of the beam electrons must decrease. In fact, as a result of the formation of the plateau on the electron distribution function, a group of particles is displaced towards the origin of the coordinate system in velocity space, thus leading to a reduction in the kinetic energy of the beam, i.e. to its retardation. Quasi-linear theory allows us to find this energy lost by the beam and to determine the form of the plasma oscillation spectrum in the system.

The equations of quasi-linear theory in the case under consideration have the form

$$\left.\begin{aligned} v_g \frac{\partial \varepsilon}{\partial x} &= A\varepsilon \frac{\partial f}{\partial v} \\ v \frac{\partial f}{\partial x} &= \frac{\partial}{\partial v} B\varepsilon \frac{\partial f}{\partial v} \end{aligned}\right\} \tag{5.31}$$

where v_g is the group velocity of the plasma waves, and the values of the coefficients A and B, which are independent of x, are given by (5.17). These equations may be obtained from (5.9) and (5.11) by introducing the following replacements:

$$\frac{\partial \varepsilon}{\partial t} \to \frac{\partial \omega}{\partial k}\frac{\partial \varepsilon}{\partial x} = v_g \frac{\partial \varepsilon}{\partial x}, \quad \frac{\partial f}{\partial t} \to \frac{\partial (p^2/2m)}{\partial p}\frac{\partial f}{\partial x} = v\frac{\partial f}{\partial x}$$

We can consider that the wave vector and velocity of a resonance particle are related by $\omega = kv$ where $\omega = \omega_0 + \frac{3}{2}k^2 \times \frac{T}{m\omega_0}$, ω_0 is the electron plasma frequence and T the electron temperature.

The height of the plateau on the distribution function may be determined from the conservation of the total number of resonance electrons (5.28) and from (5.29). The spectral energy density of the plasma oscillations as $x \to \infty$ may be found as follows. Substituting for $\varepsilon \partial f/\partial v$ from the first equation in (5.31) into the second equation, we obtain

$$\frac{\partial}{\partial x}\left(vf - \frac{\partial}{\partial v} BA^{-1} v_g \varepsilon\right) = 0$$

i.e.

$$\varepsilon_\infty(v) = \varepsilon_0(v) + AB^{-1}v_g^{-1}\int_{v_1}^{v} v\,(f_\infty - f_0)\,dv \tag{5.32}$$

Thus as the beam instability grows, and the 'peak' of the electron velocity distribution function becomes broader, the kinetic energy of the beam electrons is transferred to the plasma waves, but the total energy flux remains constant. Multiplying both sides of (5.32) by $2v_g$ and integrating over wave numbers $k = \omega/v$, we obtain an equation for the energy fluxes across the planes $x = 0$ and ∞:

$$\sum_k v_g 2 (\varepsilon_\infty - \varepsilon_0) + \int_{v_1}^{v_2} v \frac{mv^2}{2} (f_\infty - f_0) \, dv = 0 \qquad (5.33)$$

The theoretically predicted relaxation of unstable plasma to a state with a plateau on the distribution function is in agreement with experimental data [4, 5].

The quasi-linear theory [6, 7, 8] has been used to investigate a number of effects in plasma. These include the dynamics of the interaction of a mono-energetic electron beam with plasma [9], the increase in the energy of plasma particles as a result of diffusion in phase space under the influence of random fields of plasma waves [10] (with applications to geophysics, astrophysics [11] and laboratory plasma [12], diffusion of resonance particles in a non-uniform plasma under weakly turbulent conditions [13], damping of plasma waves of finite amplitude [14, 15], turbulent heating of plasma by a strong wave, and a number of others.

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Chapter 6

Interaction between High- and Low-frequency Oscillations

The frequencies of collective plasma oscillations in different oscillatory branches are usually quite different. Effects of the interaction between such collective oscillations may be described with the aid of a self-consistent system of equations [1]: the kinetic equation for the distribution function for high-frequency waves (in the six-dimensional space of coordinates and wave vectors) and equations of the hydrodynamic type for variations in density, velocity and pressure of matter in low-frequency oscillations. These equations may be used to study processes whose characteristic period and wavelength are considerably greater than the period and wavelength of the high-frequency oscillations, but this approach involves only the average characteristics of the high-frequency waves in the equations. The corresponding waves may then be regarded as quasiparticles [2].

The mean force acting on matter in the presence of high-frequency waves is proportional to the gradient of the energy density of these waves. The force acting on an individual quasi-particle is proportional to the quasi-particle

frequency gradient, i.e. the gradient of the density of matter.

The equations describing the interaction of quasi-particles with matter may be obtained from the Langrangian for a system consisting of quasi-particles and matter:

$$L = L_1 + L_2 + L_{12} \tag{6.1}$$

where L_1 is the Lagrangian for matters, e.g. for longitudinal sound oscillations in an isotropic medium $L_1 = \frac{\rho}{2}\left[\left(\frac{\partial \xi}{\partial t}\right)^2 - c_s^2 (\nabla \xi)^2\right]$, where ρ is the density of the medium, c_s is the velocity of sound, ξ is the displacement of matter, L_2 is the quasi-particle Lagrangian and L_{12} is the interaction Lagrangian. In the case of interaction with longitudinal sound waves

$$L_{12} = \sum_i s \nabla \xi (x_i)$$

where s is the 'acoustic charge' of the quasi-particle and the summation extends over all quasi-particles. Varying (6.1) over the quasi-particle coordinates x_i, we obtain the force acting on a quasi-particle:

$$\mathfrak{f} = s\nabla (\nabla \xi) \tag{6.2}$$

so that the Liouville equation for the quasi-particle distribution function N_k is of the form

$$\frac{\partial N_k}{\partial t} + \mathbf{v}_k \cdot \nabla N_k + s\nabla (\nabla \xi) \frac{\partial N_k}{\partial \mathbf{k}} = 0 \tag{6.3}$$

where $\mathbf{v}_k = \frac{\partial \omega}{\partial \mathbf{k}}$. Further, varying (6.1) over the field ξ, we obtain the homogeneous D'Alembert equation for the displacement ξ

$$\rho\left(\frac{\partial^2 \xi}{\partial t^2} - c_s^2 \nabla (\nabla \xi)\right) = -s\nabla \sum_k N_k \tag{6.4}$$

Equations (6.3) and (6.4) comprise the desired self-consistent system describing the interaction of high-frequency oscillations (quasi-particles) with the field of low-frequency waves.

As an example, consider in more detail an isotropic plasma with high electron temperature. The high-frequency oscillations are then of the Langmuir type, whereas the

low-frequency oscillations form ion acoustic waves. To find the acoustic charge of the Langmuir plasmons, s, we shall suppose that the force $\mathbf{f}$ acting on a Langmuir plasmon is

$$\mathbf{f} = -\nabla(\omega + \mathbf{k}\cdot\mathbf{u}) \tag{6.5}$$

where $\mathbf{u}$ is the velocity of matter. The velocity $\mathbf{u}$ is related to the variation in matter density δn by $u = v_\Phi \frac{\delta n}{n}$, $v_\Phi = \frac{\Omega}{q} \ll \frac{\omega}{k}$ (this is a consequence of the continuity equation for matter). Thus in (6.3) the term relating to the Doppler effect is much smaller than that connected with the density variation and, consequently, the force acting on a Langmuir plasmon is proportional to the gradient of the matter density

$$\mathbf{f} = -\frac{1}{2}\frac{\omega_0}{n}\nabla n \tag{6.6}$$

so that

$$s = \frac{1}{2}\omega_0 \tag{6.7}$$

The force acting on matter due to the gas of Langmuir waves may be found independently from the equations of motion for ions

$$M\frac{d\mathbf{u}_i}{dt} = e\mathbf{E} - \frac{\nabla p_i}{n} \tag{6.8}$$

and electrons

$$m\frac{d\mathbf{u}_e}{dt} = -e\mathbf{E} - \frac{\nabla p_e}{n} - \nabla\Phi \tag{6.9}$$

where

$$\Phi = +\frac{1}{n}\sum_{\mathbf{k}}\frac{E_{\mathbf{k}}^2}{8\pi} = \frac{1}{n}\sum_{\mathbf{k}}\frac{N_{\mathbf{k}}\omega_0}{2} \tag{6.10}$$

is the high-frequency potential. Neglecting electron inertia, we obtain from (6.8) to (6.10)

$$M\frac{d\mathbf{u}}{dt} = -\frac{\nabla(p_e + p_i)}{n} - \nabla\Phi \tag{6.11}$$

Substituting $\frac{\partial\xi}{\partial t} = \mathbf{u}$ and using the matter continuity equation

$\delta n = -n\nabla\xi$, we arrive again at (6.4) with $s = \frac{1}{2}\omega_0$. Equations (6.3) and (6.4) describe the interaction of a gas of Langmuir plasmons with the field of ion-sound waves. We shall consider several effects with the aid of the system (6.3) and (6.4).

1. Damping of ion-sound waves in a gas of Langmuir plasmons. We shall find the damping of ion-sound waves (all wave parameters will be assumed to vary as $e^{-i\Omega t+iqx}$), due to interactions with a gas of Langmuir plasmons. For this, we linearise the kinetic equation (6.3) and determine the correction to the plasmon distribution function

$$\delta N_k = \frac{is}{(\Omega - \mathbf{q}v_k)}(\mathbf{q}\cdot\xi)\,\mathbf{q}\cdot\frac{\partial N_k}{\partial \mathbf{k}} \tag{6.12}$$

Substituting δN_k in (6.4), we obtain the dispersion relation connecting the frequency Ω and the wave vector $\mathbf{q}$ of an ion-sound wave

$$-\Omega^2 + q^2c_s^2 = \frac{s^2}{\rho}q^2\int\frac{d\mathbf{k}}{\Omega - \mathbf{q}.\mathbf{v}_k}\mathbf{q}\cdot\frac{\partial N_k}{\partial \mathbf{k}} \tag{6.13}$$

Assuming that the damping (growth) rate Γ is small compared with the frequency qc_s, and substituting in (6.13)

$$\frac{1}{\Omega - \mathbf{q}.\mathbf{v}_k} = \mathcal{P}\frac{1}{\Omega - \mathbf{q}\mathbf{v}_k}\quad i\pi\delta(\Omega - \mathbf{q}.\mathbf{v}_k)$$

we find

$$\Omega = \pm qc_s + i\Gamma$$

$$\Gamma = \pi\frac{s^2q}{\rho c_s}\int\mathbf{q}\cdot\frac{\partial N}{\partial \mathbf{k}}\delta(\pm qc_s - \mathbf{q}.\mathbf{v}_k)\,d\mathbf{k} \tag{6.14}$$

where for Langmuir plasmons $s^2 = \omega_0^2/4$,

$$\mathbf{v}_k = \partial\omega/\partial\mathbf{k} = 3\mathbf{k}\omega_0 D^2$$

2. Instability of a cold gas of Langmuir plasmons. Consider small oscillations of a plasma containing a gas of Langmuir plasmons. To analyse these oscillations we use the system of hydrodynamic equations for plasmons (i.e. a system of equations connecting the moments of the distribution function N_k), which may be derived from the kinetic equation (6.3). Integrating (6.3) over $\mathbf{k}$, we obtain the continuity equation

$$\frac{\partial N}{\partial t} + \nabla N\mathbf{v} = 0 \tag{6.15}$$

where $\mathbf{v} = \int \mathbf{v}_k N_k \, d\mathbf{k} \,/ \int N_k \, d\mathbf{k}$, $N = \int N_k \, d\mathbf{k}$ and $\mathbf{v}$ is the mean plasmon velocity. Further, multiplying (6.3) by $\mathbf{v}_k$ and integrating over $\mathbf{k}$ we find the hydrodynamic equation of motion for a gas of plasmons

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = \frac{1}{m^*}\,\mathbf{f} - \frac{1}{N}\,\nabla N \langle \delta v^2 \rangle \tag{6.16}$$

where $1/m^* = 3\omega_0 D^2$ is the 'effective plasmon mass' (D is the Debye radius) and

$$\langle \delta v^2 \rangle = \frac{1}{N}\int (\mathbf{v}_k - \mathbf{v})^2 N_k \, d\mathbf{k} \tag{6.17}$$

is the mean square spread of the group velocities in the plasmon gas. For 'cold' plasmons $\langle \delta v^2 \rangle = 0$ and linearising (6.14), (6.15) and (6.16) we then have

$$\begin{aligned} &\rho(-\Omega^2 + q^2 c_s^2)\,\xi = -\frac{\omega_0}{2}\, iq\delta N \\ &-i\Omega\delta N + iqvN = 0 \\ &\qquad -i\Omega v = -\frac{1}{2m^*}\,\omega_0 q^2 \xi \end{aligned} \tag{6.18}$$

The dispersion relation for Ω and q can be found from (6.18):

$$\left(\frac{\Omega}{q}\right)^4 - c_s^2\left(\frac{\Omega}{q}\right)^2 - \beta = 0; \quad \beta = \frac{\omega_0^2 N}{4\rho m^*}$$

when $\beta \ll c_s^4$,

$$\Omega_{1,2} \simeq \pm c_s q, \quad \Omega_{3,4} \simeq \pm \frac{i}{\sqrt{2}}\,\frac{\sqrt{\beta}}{c_s}\, q \simeq \pm i \left(\frac{\sum\limits_k E_k^2}{8\pi n m}\right)^{1/2} \cdot q \tag{6.19}$$

It follows from (6.19) that a cold plasmon gas is unstable with respect to separation of a homogeneous distribution of plasmons into separate bunches.

If the effective quasi-particle mass m^* is not positive (as for Langmuir plasmons) but negative (as for ion-sound waves), the solution

$$\Omega_{3,4} \simeq \pm \frac{1}{\sqrt{2}}\,\frac{\sqrt{-\beta}}{c_s}\, q \tag{6.20}$$

defines 'slow sound' in matter containing plasmons.

3. For a 'hot' gas of Langmuir plasmons, moving as a whole relative to matter with a velocity U, we obtain the following dispersion relation from the hydrodynamic equations (6.4), (6.15) to (6.17):

$$\left(\frac{\Omega}{q}\right)^2 - \frac{\beta}{\left(\frac{\Omega}{q} - U\right)^2 - c_1^2} = c_s^2 \tag{6.21}$$

where $c_1{}^2 = \gamma\langle\delta v^2\rangle$ and γ is the effective 'adiabatic exponent' for the plasmon gas. It follows from (6.21) that the stability criterion for a gas of Langmuir plasmons at rest as a whole is of the form $c_s^2 c_1^2 > \beta$, i.e.

$$\langle\delta k^2\rangle D^2 < \frac{N\omega_0}{nT} \simeq \sum_k E_k^2 / 8\pi nT$$

where $\langle\delta k^2\rangle^{1/2}$ is the wave-vector spread in the plasmon gas.

4. Interaction of a plasmon with the random field of ion-sound waves. To find the diffusion of a Langmuir plasmon in wave number space under the influence of the random field of ion-sound waves, it is necessary to take into account the non-linear terms in the Liouville equation (6.3) for the distribution function N_k.

Separating the distribution function for Langmuir plasmons into an oscillating part (with the frequency of ion-sound waves) and a slowly varying 'background' $\langle N_k\rangle$, and averaging (6.3) over the phases of the ion-sound waves, we obtain

$$\frac{d}{dt}\langle N_k\rangle = \frac{\partial}{\partial k_\alpha} D_{\alpha\beta} \frac{\partial\langle N_k\rangle}{\partial k_\beta} \tag{6.22}$$

which is analogous to the diffusion equation in quasi-linear plasma theory (Chapter 5). The diffusion coefficient $D_{\alpha\beta}$ is defined by

$$D_{\alpha\beta} = \sum_q s^2 q_\alpha q_\beta |\mathbf{q}\cdot\boldsymbol{\xi}|^2 \,\delta(\Omega - \mathbf{q}\cdot\mathbf{v}_k) \tag{6.23}$$

In tenuous plasma it is sometimes valuable to consider separately the interaction of high-frequency oscillations with electrons and ions. For example, in the presence of Langmuir oscillations the gradient of the high-frequency potential, $f = -\nabla\Phi$, where

$$\Phi = \frac{1}{n}(E^2/8\pi) = \frac{1}{n}\sum_k \frac{E_k^2}{8\pi}$$

acts on the electrons. In this case the slow variation (compared with the period of the Langmuir oscillations) in the electron distribution function F is described by

$$\frac{\partial F}{\partial t} + \mathbf{v}\frac{\partial F}{\partial \mathbf{x}} + \frac{1}{m}\left(e\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}+\mathbf{f}\right)\frac{\partial F}{\partial \mathbf{v}} = 0 \tag{6.25}$$

Changes in the energy density spectrum of the Langmuir oscillations are determined by the equation for the plasmon distribution function $N_{\mathbf{k}} = E_{\mathbf{k}}^2/\omega_{\mathbf{k}}$:

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + v_{\mathbf{k}}\frac{\partial N_{\mathbf{k}}}{\partial \mathbf{x}} + \frac{\partial \omega_{\mathbf{k}}}{\partial \mathbf{x}}\frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}} = 0 \tag{6.26}$$

Equations (6.24) to (6.26), together with the Maxwell equations and the kinetic equation for the ion distribution function, describe slow processes in plasma in which Langmuir oscillation have been excited.

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Chapter 7

Plasmon-Plasmon Interactions

As the energy of waves in weakly turbulent plasma increases, their interaction becomes considerable. Since many waves are simultaneously excited in turbulent plasma, and since their phases are random, the interaction between the waves can be reduced to 'collisions' between them, and may be described on the basis of a kinetic equation for the wave (plasmon) distribution function in phase space.

The plasmon-plasmon interaction is analogous to the phonon-phonon interaction in condensed media, as are the corresponding kinetic equations.

THREE-PLASMON PROCESSES

Let us first consider wave interactions in which three plasmons take part. Possible processes of this type are (1) decay of one plasmon into two and (2) combination of two plasmons into one.

The frequency ω and wave vector $\mathbf{k}$ must be conserved in such processes, since otherwise the amplitude of the

transition probability will vanish. For the above two cases this yields

$$1.\ \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3; \quad \omega_1 = \omega_2 + \omega_3 \tag{7.1}$$

$$2.\ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3; \quad \omega_1 + \omega_2 = \omega_3 \tag{7.2}$$

In isotropic plasma, in which there are two types of longitudinal plasmon, i.e. ion sound (s) and Langmuir (l) oscillations, the conservation equations (7.1) and (7.2) allow only those three-plasmon interactions in which two Langmuir and one ion-sound plasmon participate (we are assuming that the necessary condition for the existence of weakly damped ion-sound waves is fulfilled, i.e. the electron pressure greatly exceeds the ion pressure and, moreover, we are considering only weakly damped long-wavelength Langmuir oscillations). In fact, long-wavelength l-plasmons have approximately equal frequencies $\omega = \omega_0$, so that one l-plasmon cannot decay into two, and conversely, two l-plasmons may not combine to form one. Ion-sound oscillations may not interact among themselves in three-plasmon processes, because the frequency ω of these oscillations increases with wave number k less rapidly than in accordance with the linear law. Finally, three-plasmon processes involving two s-plasmons and one l-plasmon are impossible in a plasma with sufficiently small ratio of electron to ion masses, since the maximum possible ion-sound wave frequency is then considerably lower than that of the Langmuir oscillations, and the frequency may not be conserved in such processes.

By considering the only possible three-plasmon processes, i.e. those involving one s-plasmon and two l-plasmons, we can, using the method of second quantisation, write a system of kinetic equations for the distribution function of l- and s-plasmons. For the l-plasmon distribution function N_k we have

$$\begin{aligned} \dot{N}_1 &= \sum w_{1,2}\left(-N_1(N_3+1)(n_2+1)+(N_1+1)N_3 n_2\right) \\ &\quad + \sum w_{2,1}\left(-N_1(N_3+1)n_2+(N_1+1)N_3(n_2+1)\right) \end{aligned} \tag{7.3}$$

where $w_{1,2}$ is the probability of the decay of an l-plasmon $\mathbf{k}_1$ into an s-plasmon $\mathbf{k}_2$ and an l-plasmon $\mathbf{k}_3$, and $w_{2,1}$ is the probability of combination of an l-plasmon $\mathbf{k}_1$ and an

s-plasmon $\mathbf{k}_2$ into an l-plasmon $\mathbf{k}_3$. The summation in (7.3) is taken over the wave numbers $\mathbf{k}_2$ and $\mathbf{k}_3$, satisfying the conservation laws (7.1) in the first and (7.2) in the second collision integral.

The analogous equations for the s-plasmon distribution function n_k are

$$\dot{n}_2 = \sum w_{2,1}\left(-n_2 N_1 (N_3+1) + (n_2+1)(N_1+1) N_3\right) \qquad (7.4)$$

where $w_{2,1}$ is the probability of combination of an s-plasmon $\mathbf{k}_2$ and an l-plasmon $\mathbf{k}_1$ into an l-plasmon $\mathbf{k}_3$. The second collision integral in (7.4) disappears, since the decay of a low-frequency s-plasmon into two high-frequency l-plasmons is forbidden by the frequency conservation law (as in the inverse process of formation of one s-plasmon from two l-plasmons).

The structure of the collision integrals (7.3) and (7.4) is somewhat simpler in the case of classical plasma. In fact, since the mean 'occupation number' N_k is related to the spectral energy density of the oscillations Q_k by $N_k = Q_k/\hbar\omega_k$, e.g. for long-wave Langmuir oscillations, it follows that the case of classical plasma ($\hbar \to 0$) we must retain only the highest-order terms N or n in the collision integrals. The collision integrals (7.3) and (7.4) are then quadratic in N and n:

$$-N_1 n_2 (N_3+1) + (N_1+1)(n_2+1) N_3 \xrightarrow[\hbar \to 0]{} -n_2 N_1 + n_2 N_3 + N_1 N_3$$

$$-N_1 (n_2+1)(N_3+1) + (N_1+1) n_2 N_3 \xrightarrow[\hbar \to 0]{} -N_1 n_2 - N_1 N_3 + n_2 N_3$$

and after the substitution $N_k = Q/\hbar\omega_k$, Planck's constant disappears from (7.3) and (7.4).

The probabilities of the various processes of interaction between waves (and also between waves and particles) in weakly turbulent plasma, may be found in various ways, e.g. by the method of random phases in the hydrodynamic equations of motion for a plasma [1, 2], the Lagrangian method [3, 4], the method of correlation functions [5, 6], by averaging the kinetic equations [7], the diagram method [8] and so on. We will derive the kinetic equations for a system of Langmuir and ion-sound plasmons in isotropic plasma from the self-consistent system of Equations (6.3) and (6.4) which describe the interaction of a gas of Langmuir plasmons with ion-sound waves. At the same time we will obtain an expression for the probability of emission and absorption

of ion-sound waves by Langmuir plasmons when the wave vector $\mathbf{q}$ of the ion-sound wave is much smaller than the wave vector $\mathbf{k}$ of the l-plasmon.

The kinetic equation (7.4) for ion-sound wave (s-plasmon) distribution function, taking into account the emission and absorption of these waves by l-plasmons, is of the form

$$\dot{n}_q = n_q \sum_k w\mathbf{q} \cdot \frac{\partial N}{\partial \mathbf{k}} \delta(\Omega_q - \mathbf{k}\mathbf{v}_k)$$

where $q/k \ll 1$.
Comparing the quantity $\Gamma = \dot{n}_q/2n_q$ with (6.14), which applies to the case of Langmuir plasmons, we find the probability of emission and absorption of a long-wave s-plasmon by an l-plasmon:

$$w = \frac{\pi}{4} \frac{\omega_0^2}{nMc_s} q \tag{7.5}$$

Similarly, Equation (6.22), describing the diffusion of plasmons in the random field of ion-sound waves, may be obtained independently of the kinetic equation (7.3) for the l-plasmon distribution function, which accounts for the processes of emission and absorption of ion-sound waves by Langmuir plasmons. In fact, if the occupation number of ion-sound waves is much greater than the occupation number of the Langmuir oscillations, the rate of change of the Langmuir plasmon distribution function, due to the direct and inverse processes of emission and absorption of ion-sound waves is

$$\begin{aligned} \dot{N}_\mathbf{k} = \sum_\mathbf{q} [&w_{\mathbf{k},\mathbf{k}-\mathbf{q}} n_\mathbf{q} (N_{\mathbf{k}-\mathbf{q}} - N_\mathbf{k}) \delta(\omega_\mathbf{k} - \omega_{\mathbf{k}-\mathbf{q}} - \Omega_\mathbf{q}) \\ &+ w_{\mathbf{k},\mathbf{k}+\mathbf{q}} n_\mathbf{q} (N_{\mathbf{k}+\mathbf{q}} - N_\mathbf{k}) \delta(\omega_\mathbf{k} - \omega_{\mathbf{k}+\mathbf{q}} + \Omega_\mathbf{q})] \end{aligned} \tag{7.6}$$

Substituting

$$\Psi_{\mathbf{k},\mathbf{q}} = w_{\mathbf{k},\mathbf{k}-\mathbf{q}} n_\mathbf{q} (N_\mathbf{k} - N_{\mathbf{k}-\mathbf{q}}) \delta(\omega_\mathbf{k} - \omega_{\mathbf{k}-\mathbf{q}} - \Omega_\mathbf{q})$$

we can write the right-hand side of (7.6) in the form

$$\dot{N}_\mathbf{k} = \sum_\mathbf{q} (\Psi_{\mathbf{k}+\mathbf{q},\mathbf{k}} - \Psi_{\mathbf{k},\mathbf{q}}) n_\mathbf{q}$$

Expanding this expression in powers of q, for $q/k \ll 1$, we

obtain

$$\dot{N}_{\mathbf{k}} = \frac{\partial}{\partial \mathbf{k}_\alpha} D_{\alpha\beta} \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}_\beta}$$

where diffusion coefficient $D_{\alpha\beta}$ is related to probability w by

$$D = \sum_q n_q q_\alpha q_\beta w_{\mathbf{k},\,\mathbf{k}-\mathbf{q}} \delta(\Omega - \mathbf{q}\cdot\mathbf{v}_k)$$

Substituting for w from (7.5), we obtain Equations (6.22) and (6.23).

The characteristic 'collision frequency' of s-plasmons with l-plasmons (for $k \sim D^{-1}$) may be found by substituting $w \approx \frac{\omega_0^2}{nMc_s}$ and $n_k \approx \frac{|E_k^l|^2}{\omega_0}$ in (7.4):

$$\gamma_3 = \frac{\dot{n}_2}{n_2} \approx \Omega \frac{\langle |E^l|^2 \rangle}{nT}$$

It is useful to write the expression for γ_3 in the form

$$\gamma_3 \approx \Omega \left(\frac{v_\sim}{U}\right)^2$$

where $v_\sim$ is the random velocity of the electrons in Langmuir oscillations and U is the phase (or group) velocity of of the Langmuir waves with $k \sim D^{-1}$.

The quantity γ_3 determines a number of properties of weakly turbulent plasma. For example, the characteristic damping length L for ion-sound waves in plasma with Langmuir oscillations is approximately equal to the mean free path of an s-plasmon before collision with an l-plasmon and, consequently, is related to the collision rate γ_3 by $L \approx c_s/\gamma_3$.

In addition to the processes of emission and absorption of an ion-sound wave by a Langmuir plasmon, the three-plasmon process of combination and decay may take place with longitudinal and transverse plasmons in isotropic plasma. For example, a sufficiently short-wavelength transverse plasmon ($\omega = \sqrt{\omega_0^2 + c^2k^2}$) may emit Langmuir waves ($\omega \simeq \omega_0$) [11] or ion-sound waves; two Langmuir waves ($\omega = \omega_0$) may combine into one transverse wave with frequency $2\omega_0$ [10] and so on.

Three-plasmon processes in anisotropic and non-uniform plasma determine many of the properties of turbulent plasma, in particular, the coefficient of turbulent transport of matter, momentum and energy. The values of these coefficients are needed to obtain the solutions of a number of

problems; including the structure of the turbulent front of a shock wave in tenuous plasma [1, 2], 'anomalous' diffusion in plasma [9] and so on.

FOUR-PLASMON PROCESSES

In a number of cases, the dependence of the frequency of collective plasma oscillations on the wave vector is such that interaction processes involving three plasmons are forbidden by the laws of conservation of frequency and wave number, and it is necessary to consider processes involving four waves.

In the case of isotropic plasma, which we will consider below, the conservation laws forbid three-plasmon processes in which only Langmuir waves or only ion-sound oscillations take part. For this reason the kinetics of electron Langmuir oscillations excited in the plasma, are determined by an equation which takes into account processes involving the interaction of four Langmuir waves as well as the three-plasmon processes considered above. If, however, ion-sound oscillations are excited in the plasma, their kinetic properties are determined by the four-wave interaction processes.

In general, the following four-plasma processes are possible: (1) transformation of two plasmons into two, (2) decay of one plasmon into three, (3) combination of three plasmons into one. Thus the kinetic equation for the waves, taking only four-plasmon processes into account, is of the form

$$\begin{aligned}\dot{N}_1 = {} & \sum w_{2,2}\left((N_1+1)(N_2+1)N_3N_4 - N_1N_2(N_3+1)(N_4+1)\right)\\ & + \sum w_{1,3}\left((N_1+1)N_2N_3N_4 - N_1(N_2+1)(N_3+1)(N_4+1)\right)\\ & + \sum w_{3,1}\left((N_1+1)(N_2+1)(N_3+1)N_4 - N_1N_2N_3(N_4+1)\right)\end{aligned} \tag{7.7}$$

The summation in (7.7) extends over the wave numbers k_2, k_3, k_4, taking into account the conservation laws which for the first, second and third terms of the right-hand side of (7.7) have the following forms, respectively,

1. $k_1 + k_2 = k_3 + k_4, \quad \omega_1 + \omega_2 = \omega_3 + \omega_4$
2. $k_1 = k_2 + k_3 + k_4, \quad \omega_1 = \omega_2 + \omega_3 + \omega_4$
3. $k_1 + k_2 + k_3 = k_4, \quad \omega_1 + \omega_2 + \omega_3 = \omega_4$

For ion-sound oscillations the kinetic equation contains all three terms. Processes (2) and (3) are forbidden for long-wavelength electron Langmuir oscillations (since for these oscillations the frequencies are approximately equal to ω_0), so that (7.7) takes the form

$$\dot{N}_1 = \sum w_{2,2}((N_1+1)(N_2+1)N_3N_4 - N_1N_2(N_3+1)(N_4+1)) \tag{7.8}$$

The collision integral for four-plasmon processes in the kinetic equation for Langmuir oscillations is therefore

$$\dot{N} = \int w((N_1+1)(N_2+1)N_3N_4 - N_1N_2(N_3+1)(N_4+1)) \times \delta(\omega_1+\omega_2-\omega_3-\omega_4)\,dk_2\,dk_3 \tag{7.9}$$

where $k_4 = k_1 + k_2 - k_3$ and $w \approx \frac{\hbar^2 k^4}{n^2 m^2}$.

From Equation (7.9), substituting $N_k = E_k^2/4\pi\hbar\omega_0$, we find the characteristic collision rate in a gas of Langmuir waves:

$$\gamma_4 = \frac{\dot{N}}{2N} \approx \omega\left(\frac{E^2}{nT}\right)^2 \tag{7.10}$$

where $\omega \approx \omega_0$ and $E^2/4\pi = \sum\limits_k E_k^2/4\pi$ is the energy density of the oscillations. The collision rate in a gas of waves can be calculated as follows. Since the collision integral is proportional to N^3, the rate γ_4 should be proportional to N^2, i.e. to the fourth power of the ratio of the amplitude of the random velocity of electrons in the Langmuir oscillations to the phase (or group) velocity of these oscillations (with wave number $k \sim D^{-1}$). For the dimensions to be correct, the coefficient of proportionality must be equal to the oscillation frequency, so that

$$\gamma_4 \approx \omega\left(\frac{v_\sim}{U}\right)^4 \tag{7.11}$$

Substituting $v_\sim \approx \frac{eE}{m\omega_0}$, $U \approx \sqrt{T/m}$, we again obtain Equation (7.10).

The rate γ_4 determines a number of properties of turbulent plasma. The decay time of the turbulent spectrum of Langmuir oscillations excited in plasma is approximately γ_4^{-1}. The rate γ_4 (or the mean free path for Langmuir waves

colliding among themselves) determines the energy flux $\mathbf{Q}$ in a gas of Langmuir waves, $\mathbf{Q} \approx \frac{v_T^2}{\gamma_4} \nabla E^2$. The rate of non-linear damping of a Langmuir wave of finite amplitude is, in order of magnitude, also equal to γ_4.

As was mentioned above, the structure of the collision integral for ion waves (among themselves) is more complicated than for Langmuir waves; the collision frequency in a gas of ion-sound waves may be evaluated with the aid of (7.11) by substituting $v_{\sim} \approx eE/M\omega_{0i}$, $U \approx \sqrt{T/M}$ and $\omega \approx \omega_{0i}$ into it:

$$\gamma_4^s \approx \omega_{0i} \left(\frac{E^2}{nT}\right)^2$$

The description of the physical properties of weakly turbulent plasma may require the consideration of not only three- and four-plasmon processes, but also processes involving a greater number of waves, as well as plasmon-particle scattering (the plasmons in the initial and final states may belong to the same or to different oscillatory branches) or the simultaneous emission or absorption of two plasmons by a particle. The need to consider this last process may arise, for example, when the laws of conservation of frequency and wave number forbid the emission or absorption of a single plasmon by a particle.

For example, the collision term in the kinetic equation for plasmons 1, describing plasmon scattering by particles, is of the form

$$\dot{N}_1 = \sum w \left(-N_1 (N_2 + 1) f_3 (1 - f_4) + (N_1 + 1) N_2 (1 - f_3) f_4\right) \tag{7.12}$$

(the summation is performed subject to the wave-vector conservation law $q_1 + k_3 = q_2 + k_4$, $\omega_1 + \varepsilon_k = \omega_2 + \varepsilon_4$; $N_{1,2}$ is the plasmon distribution function and f_k the particle distribution function). The contribution of plasmon-particle scattering to the collision term in the kinetic equations for the distribution function of plasmons 2 and to the particle distribution function can be found in a similar way.

The probabilities of plasmon-particle scattering and of two-plasmon emission and absorption by particles are calculated in [7, 8, 12].

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Chapter 8

Strong Turbulence

Kinetic equations such as (7.3), (7.4), (7.7) and (7.12) cannot be used in the study of strongly turbulent plasma, since the interaction between waves is then stronger, and turbulence cannot be described in terms of occupation numbers for these waves (normal modes in a laminar plasma). However, for strongly developed turbulence it is often possible to evaluate the intensity of the pulsations and to find their spectral density by recalling that in extremely turbulent states the amplitude of the particle velocity in the pulsations is of the same order as the phase velocity of the waves whose growth led to the given turbulent state.

The physical basis for this is as follows. As waves of finite amplitude propagate in laminar plasma, the character of the wave motion changes abruptly when the amplitude of the velocity $v_\sim$ of the particles in the wave becomes comparable in order of magnitude with the phase velocity of the wave ω_k/k. When $v_\sim$ exceeds ω_k/k, the wave becomes unstable and strong dissipation of the wave energy sets in [1]. As a rule, the maximum possible wave amplitude in laminar plasma is therefore of the same order of magnitude

as the phase velocity of the wave. We may assume that strong dissipation will also occur in turbulent plasma for $v_{\sim} > \omega_k/k$, in spite of the interactions of separate waves with the turbulent 'background' of all the other pulsations. The velocity amplitude of each wave in extremely turbulent plasma should then be approximately equal to the phase velocity

$$\langle v_{\sim}^2 \rangle^{1/2} \approx \frac{\omega_k}{k} \tag{8.1}$$

We shall use (8.1) to consider a number of examples.

1. From (8.1), and remembering that, in the three-dimensional case $v_{\sim}^2 \approx v_k^2 k^3$, we obtain for a plasma with an isotropic distribution of strongly excited Langmuir oscillations:

$$v_k \approx v_{\sim} k^{-3/2} \approx \omega_0 k^{-5/2} \tag{8.2}$$

and thus, the spectral energy density of pulsations is

$$nmv_k^2 \approx \frac{E_k^2}{4\pi} \approx nm\omega_0^2 k^{-5} \tag{8.3}$$

2. Ion-sound oscillations. For an isotropic distribution of strongly excited ion-sound waves (we limit ourselves to the case $kD \ll 1$) we have

$$v_k \approx v_{\sim} k^{-3/2} \approx c_s k^{-3/2} \tag{8.4}$$

so that the energy-density spectrum is equal to

$$nMv_k^2 \approx nMc_s^2 k^{-3} \tag{8.5}$$

and the frequency spectral density of the potential is given by

$$\varphi_\omega^2 = \varphi_k^2 \frac{4\pi k^2\, dk}{d\omega} \approx \frac{T^2}{e^2} \frac{1}{\omega} \tag{8.6}$$

3. Turbulent current-carrying plasma. Consider a plasma state which is strongly turbulent arising from the growth of perturbations during the flow of an electric current through the plasma (beam instability). Langmuir oscillations of the electron component of the plasma are then the most strongly excited. To evaluate the amplitude of the turbulent pulsations, we must transform from the laboratory coordinate system (in which the centre of mass of the plasma is at

rest) to a coordinate system in which the electron gas as whole is at rest. In this coordinate system, the linear theory of beam instability (Chapter 3) shows that oscillations with $\omega/k \approx U$ grow most rapidly (U is the relative velocity of the ion and electron gases). Consequently, in accordance with (8.1), the mean random velocity of electrons in the turbulent pulsations should be of the same order of magnitude as the velocity U of the current

$$\langle v_{\sim}^2 \rangle^{1/2} \approx U \tag{8.7}$$

Where a current flows through a plasma in a tube of radius a, the effective conductivity of the turbulent plasma and the amplitude of the turbulent pulsations may be found from the energy balance equation. The energy transferred to a unit volume of the electron gas per unit time by the electric field is equal to $neEU$; the rate of loss of energy from a unit volume as a result of the Langmuir plasmons leaving through the walls of the tube is equal to $\langle E_{\sim}^2 \rangle/\tau$, where τ is the characteristic time for a Langmuir plasmon to leave through the walls ($\tau \sim a/v$; v is the average group velocity of the plasmons). Equating these quantities and substituting $\langle E_{\sim}^2 \rangle \approx nm\langle v_{\sim}^2 \rangle \approx nmU^2$ in accordance with (8.7), we obtain

$$U = \frac{e\tau}{m} E \tag{8.8}$$

Experimental results [2] agree qualitatively with the dependence given by (8.8).

We note that the relation given by (8.1), which is valid for strong turbulence in gravity waves on the surface of the sea, leads to a law of the form $\Phi_\omega \backsim \omega^{-5}$ for the energy distribution in the pulsation spectrum, which is in satisfactory agreement with experiment [3, 4]. In accordance with (8.1)

$$v_{\sim} \approx \frac{\omega_k}{k}$$

where $\omega_k \approx \sqrt{gk}$ (g is the acceleration due to gravity). Since gravity waves on the surface of the sea are really two-dimensional, $v_k \approx v_{\sim} k^{-1}$ and the energy density $\mathscr{E}_k$, which is equal to $\rho v_k^2 \lambda$ ($\lambda \approx \frac{1}{k}$ is the characteristic damping length, for a pulsation of wave number k), is determined by

$$\mathscr{E}_k \approx \rho g \cdot \frac{1}{k^4}$$

and

$$\mathscr{E}_\omega = \mathscr{E}_k \frac{2\pi k\, dk}{d\omega} \approx \rho g^3 \frac{1}{\omega^5}$$

For pulsations in the displacement of the sea surface, we obtain

$$\Phi_\omega = \xi^2_\omega = \xi^2_k \frac{2\pi k\, dk}{d\omega} \approx \frac{g^2}{\omega^5}$$

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Chapter 9

Transport Coefficients in Turbulent Plasma

The transport coefficients (conductivity, diffusion coefficient, etc.) of laminar and turbulent plasma in strong magnetic fields may differ greatly even for weak turbulence, when the noise level is relatively small. In laminar magnetised plasma ($\Omega\tau \gg 1$) the coefficients of transport at right angles to the magnetic field are small (of the order of $(\Omega\tau)^{-1}$ or $(\Omega\tau)^{-2}$) because the charged plasma particles are displaced during collisions through distances of the order of the Larmor radius, which is very small in a strong magnetic field. In turbulent plasma, even weak random electric and magnetic fields lead to considerably greater mean displacements of electrons and ions, i.e. greater current and diffusion flow than in laminar plasma. Turbulence may thus greatly alter the mean macroscopic transport coefficients.

ANOMALOUS DIFFUSION IN PLASMA

Random pulsations in the electric and magnetic fields in turbulent plasma may give rise to a sharp increase in the

diffusion coefficient at right angles to the magnetic field, i.e. they may lead to the appearance of 'anomalous' diffusion. We shall consider, first of all, the anomalous diffusion of resonance particles (i.e. those whose velocity $v_{\parallel}$ along the lines of force of a strong magnetic field coincides with the quantity $\omega_k/k_{\parallel}$) in weakly non-uniform plasma acted on by low-frequency electric fields [1, 2]. Substituting in the expression

$$D_{\perp} = \frac{\langle(\Delta x_{\perp})^2\rangle}{\Delta t}$$

the particle displacement arising from drift in the electric field of the pulsations

$$\Delta x_{\perp} = \int_0^{\Delta t} dt\, \frac{c}{H} E_{\perp}(z + v_{\parallel}t,\, t)$$

and averaging over the random phases of the fields

$$E_{\perp} = \sum_k E_{\perp k} e^{ik_{\parallel}z - i\omega_k t}$$

we obtain

$$D_{\perp} = \frac{c^2}{H^2} \int dk\, |E_{\perp k}|^2\, \delta(\omega_k - k_{\parallel}v_{\parallel}) \qquad (9.1)$$

The diffusion coefficient (9.1) is of the order of magnitude of $\frac{c^2}{\langle\omega\rangle}\frac{\langle E^2\rangle}{H^2}$ and may greatly exceed the 'classical value' $\frac{\nu T}{\omega_{He}^2 m}$. However, even (9.1) does not give the total diffusion coefficient for weakly turbulent plasma, i.e. it does not determine the total flux of particles at right angles to the magnetic field. To find the transverse plasma flux (in the presence of a pressure gradient) in a strong magnetic field it is necessary to know the magnitude of the momentum transferred in a unit time by the charged particles to the plasmon gas of the weakly turbulent plasma. Two limiting cases can be distinguished: (1) the plasmon gas is at rest (for example, it is restrained by friction on the walls of the container) and the electron and ion gases, moving with respect to the plasmon gas, interact with it; (2) friction

between the plasmon gas and the walls (or the neutral gas) does not exist; plasmons are emitted by one charged gas and absorbed by the other, achieving an exchange of momentum between electrons and ions (friction arises between electrons and ions).

1. Quiescent plasmon gas. We shall consider the ambipolar diffusion of plasma in a neutral cylindrical column (located in a uniform axial magnetic field H) under the action of a radial pressure gradient and in the presence of a force due to friction on the quiescent plasmon gas $F_{\pm} = - m_{\pm} \nu_{\pm} v_{\pm}$ ($m_{\pm}$, $v_{\pm}$, $\nu_{\pm}$ are the mass, velocity and effective rate of collision with plasmons for the ions and electrons respectively). The azimuthal components of the equations of motion for the electrons and ions

$$-\frac{\nabla p_{\pm}}{n} + e_{\pm}\mathbf{E} + e_{\pm}\frac{\mathbf{v}_{\pm}}{c} \times \mathbf{H} + \mathbf{F}_{\pm} = 0 \qquad (9.2)$$

yield

$$\frac{e_{\pm}}{c} v_{r\pm} H - m_{\pm}\nu_{\pm} v_{\theta\pm} = 0 \qquad (9.3)$$

Substituting for $v_{\theta\pm}$ from (9.3) in the equation obtained by summing the radial components of (9.2)

$$-\frac{\nabla (p_{+} + p_{-})}{n} + \frac{e}{c}(v_{\theta+} - v_{\theta-}) H - (m_{+}\nu_{+} + m_{-}\nu_{-}) v_{r} = 0 \qquad (9.4)$$

we find the radial flow of plasma (assuming that $\nabla p_{\pm} = T\nabla n$) and the diffusion coefficient $D = nv_r/|\nabla n|$:

$$D = \frac{2T}{m_{+}\nu_{+} + m_{-}\nu_{-}} \frac{1}{1 + \left(\frac{eH}{c}\right)^2 \frac{1}{m_{+}\nu_{+}} \frac{1}{m_{-}\nu_{-}}} \qquad (9.5)$$

The maximum value of (9.5) for arbitrary positive values of $m_{+}\nu_{+}$ and $m_{-}\nu_{-}$ is

$$D_{\max} = \frac{1}{2} \frac{cT}{eH} \qquad (9.6)$$

The maximum possible coefficient of anomalous diffusion for a quiescent plasmon gas is the same as the well-known Bohm diffusion coefficient [3], (Compare this with [4], which agrees with some experimental data [5,6]).

2. Collective friction between electrons and ions, arising from induced emission or absorption of plasmons, may be

taken into account approximately, by setting in (9.2)

$$\mathbf{F}_{\pm} = \mp m_{-}\nu(\mathbf{v}_{+} - \mathbf{v}_{-}) \tag{9.7}$$

where ν is the 'effective collision rate' (direct calculation of the frictional force between electrons and ions in weakly turbulent plasma may be carried out by solving the system of kinetic equations for plasmons and particles). The velocity of radial motion of the plasma, arising from the radial pressure gradient $\nabla(p_{+} + p_{-}) = 2T\nabla n$, is

$$v_r = 2\frac{\nu}{\omega_{He}}\frac{cT}{eH}\frac{\nabla n}{n} \tag{9.8}$$

and the corresponding diffusion coefficient $D = nv_r/|\nabla n|$ is

$$D = 2\frac{\nu}{\omega_{He}}\frac{cT}{eH} \tag{9.9}$$

The 'effective collision rate' ν and the coefficient of anomalous diffusion (9.9) increase monotonically with increase in the amplitude of the turbulent pulsations. When the collision rate becomes much greater than the cyclotron frequency, ambipolar diffusion of the plasma is replaced by flow in which inertial forces of the spreading plasma are balanced by the pressure gradient.

ELECTRICAL CONDUCTIVITY OF WEAKLY TURBULENT PLASMA

The influence of steady-state turbulent noise on the electrical conductivity is most readily determined in the case of a magnetised, weakly ionised plasma in which collisions of electrons and ions with the neutral particles play a leading role. Let us assume that the frequency ω of the turbulent pulsations is lower than the collision rate ($\omega\tau \ll 1$) and that the characteristic length of the pulsations greatly exceeds the average Larmor radius of ions and electrons. We can then use the generalised Ohm's law

$$j_{\alpha} = \sigma_{\alpha\beta}E_{\beta} \tag{9.10}$$

which is valid at every point in space, as a basic equation for finding the effective conductivity. The conductivity tensor $\sigma_{\alpha\beta}$ has the following form in a strong magnetic field ($\Omega\tau \gg 1$; $\Omega = eH/mc$):

$$\begin{pmatrix} \sigma_2 \, \sigma_1 \, 0 \\ -\sigma_1 \, \sigma_2 \, 0 \\ 0 \; 0 \; \sigma \end{pmatrix} \qquad (9.11)$$

where $\sigma_k = \sigma / (\Omega\tau)^k$ and $\sigma = \frac{ne^2}{m}\tau$ is the plasma conductivity along the lines of force of the magnetic field, which is considered to be directed along the z axis. Equation (9.11) is obtained from the equation of motion for electrons $\mathbf{j} + \mathbf{j} \times \mathbf{\Omega}\tau = \sigma\mathbf{E}$. The relation given by (9.10) connects the electric field with the current density at every point in the turbulent plasma. To find the effective conductivity, it is necessary to average (9.10) over the volume of the system [9, 10]. Choosing the x direction to be along the resultant mean current, we obtain

$$\begin{aligned} \langle j_x \rangle &= \langle \sigma_1 E_y \rangle \\ 0 &= -\langle \sigma_1 E_x \rangle + \langle \sigma_2 E_y \rangle \end{aligned} \qquad (9.12)$$

The inequalities $E_y \gg E_x$, $\sigma_1 \gg \sigma_2$, valid in a strong magnetic field, are taken into account in the averaging procedure.

To calculate the mean value of the product $\langle \sigma E \rangle$ in (9.12), it is necessary to express the field fluctuations $E = E - \langle E \rangle$ in terms of the average field $\langle E \rangle$ and the conductivity fluctuations $\tilde{\sigma} = \sigma - \langle \sigma \rangle$ which are regarded as given. Since $\sigma_1 \gg \sigma_2$ and at the same time the fluctuations $\tilde{E}_x$ and $\tilde{E}_y$ are of the same order, we may neglect fluctuations of σ_2. Taking this into account, we obtain from the generalised Ohm's law the following relation between the electric field and conductivity fluctuations:

$$\begin{aligned} \tilde{j}_x &= \langle \sigma_2 \rangle \tilde{E}_x + \langle \sigma_1 \rangle \tilde{E}_y + \tilde{\sigma}_1 \langle E_y \rangle \\ \tilde{j}_y &= \langle \sigma_1 \rangle \tilde{E}_x - \tilde{\sigma}_1 \langle E_x \rangle + \langle \sigma_2 \rangle \tilde{E}_y \end{aligned} \qquad (9.13)$$

Using the result $\nabla \cdot \mathbf{j} = \nabla \times \mathbf{E} = 0$, we find the Fourier components of the electric field

$$\tilde{\mathbf{E}}_k = i\mathbf{k} \left(\langle E_x \rangle k_y - \langle E_y \rangle k_x \right) \tilde{\sigma}_{1,\mathbf{k}} / \left(\langle \sigma_2 \rangle k_\perp^2 + \langle \sigma \rangle k_\parallel^2 \right) \qquad (9.14)$$

where $k_\parallel$ and $k_\perp$ are the longitudinal and transverse components of the wave vector k relative to the magnetic field.

From (9.12) we obtain, to the same degree of accuracy

$$\begin{aligned} \langle j_x \rangle &= \langle \sigma_1 \rangle \langle E_y \rangle \\ 0 &= -\langle \sigma_1 \rangle \langle E_x \rangle + \langle \sigma_2 \rangle \langle E_y \rangle - \sum_k \widetilde{\sigma}_{1,-k} \widetilde{E}_{xk} \end{aligned} \tag{9.15}$$

Substituting (9.14) into (9.15) we obtain the relationship between the mean electric field and the mean current density:

$$\langle j_x \rangle = \sigma_{eff} \langle E_x \rangle \tag{9.16}$$

where

$$\sigma_{eff} = \langle \sigma \rangle \left(1 + \frac{(\Omega\tau)^2}{\langle \sigma_1 \rangle^2} \sum_k \frac{|\sigma_{1,k}|^2 k_x^2}{k_\perp^2 + (\Omega\tau)^2 k_\parallel^2} \right)^{-1} \tag{9.17}$$

Replacing the sum by an integral, and integrating over the modulus of the wave vector k, we find

$$\sigma_{eff} = \langle \sigma \rangle \left(1 + (\Omega\tau)^2 \iint \frac{Q(\theta, \varphi) \cos^2 \varphi \, d\varphi \, d\cos\theta}{1 + (\Omega\tau)^2 \cos^2 \theta} \right)^{-1} \tag{9.18}$$

where $Q(\theta, \varphi)$ is the angular distribution of the spectral density of the turbulent fluctuations of density, normalised so that

$$\int Q(\theta, \varphi) \, d\varphi \, d\cos\theta = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle^2} \equiv \left\langle \left(\frac{\Delta n}{n} \right)^2 \right\rangle \tag{9.19}$$

We have used spherical coordinates with the polar axis along the magnetic field; the azimuthal angle φ is measured from the direction of the current vector. It should be noted that the effective conductivity (9.18) is determined exclusively by fluctuations in density of the ionised plasma components.

Further analysis of the expression for the effective conductivity requires the introduction of assumptions regarding the spectral density of the pulsations. As can be seen from (9.18), when $\Omega\tau \gg 1$ the effective conductivity is determined only by pulsations whose wave vector is perpendicular to the magnetic field $\left(\left| \theta - \frac{\pi}{2} \right| \lesssim (\Omega\tau)^{-1} \right)$. The result will therefore be very dependent on the degree of anisotropy of the fluctuation spectrum.

Let us consider first the case in which the function Q is sufficiently smooth (the characteristic measure of a

change in $Q(\theta)$ considerably exceeds $(\Omega\tau)^{-1}$). The integral over θ is then $\int Q(\theta, \varphi)(1+(\Omega\tau)^2\cos^2\theta)^{-1} d(\cos\theta) = (\Omega\tau)^{-1}\pi Q \times\left(\frac{\pi}{2}, \varphi\right)$ so that the effective conductivity is given by:

$$\sigma_{eff} = \sigma\left(1 + \pi\Omega\tau\int_0^{2\pi} Q\left(\frac{\pi}{2}, \varphi\right)\cos^2\varphi\, d\varphi\right)^{-1} \tag{9.20}$$

In the case under consideration, in a strong magnetic field ($\Omega\tau Q \gg 1$), the conductivity is inversely proportional to the magnetic field. We must, however, remember that the spectral density of the pulsations may itself depend on the magnetic field. In the isotropic case, we can use (9.19) to rewrite the expression for the conductivity (9.20) in the following form:

$$\sigma_{eff} = \sigma\left(1 + \frac{\pi}{4}\Omega\tau\left\langle\left(\frac{\Delta n}{n}\right)^2\right\rangle\right)^{-1} \tag{9.21}$$

Let us now consider the limiting case of a strongly non-isotropic fluctuation spectrum. In particular, we shall assume that the characteristic angle for a change in the function $Q(\theta)$ is considerably smaller than $(\Omega\tau)^{-1}$. From (9.18) we then have

$$\sigma_{eff} = \sigma\left(1 + (\Omega\tau)^2\int_0^{2\pi} P(\varphi)\cos^2\varphi\, d\varphi\right)^{-1} \tag{9.22}$$

where $P(\varphi)$ is the angular distribution of the noise spectral density, normalised so that

$$\int_0^{2\pi} P(\varphi)\, d\varphi = \left\langle\left(\frac{\Delta n}{n}\right)^2\right\rangle$$

A simpler expression is obtained for the conductivity in the case of one-dimensional fluctuations [11]

$$\sigma_{eff} = \sigma\left(1 + (\Omega\tau)^2\left\langle\left(\frac{\Delta n}{n}\right)^2\right\rangle\right)^{-1} \tag{9.23}$$

In this case the conductivity is proportional to the square of the magnetic field.

Let us consider the case of the one-dimensional spectrum in greater detail. The effective conductivity can now be calculated without assuming that the fluctuations are small or that the magnetic field is large. If all the quantities depend on only one space coordinate x (the magnetic field is parallel

to the z axis), we find from the equation $\nabla \cdot \mathbf{j} = \nabla \times \mathbf{E} = 0$, that

$$j_x,\ E_y,\ E_z = \text{const}$$

Using these conditions, we may apply an averaging procedure to Ohm's law. For example, the x component of Ohm's law

$$j_x = \sigma_1 E_y + \sigma_2 E_x$$

gives, after averaging,

$$\left\langle \frac{1}{\sigma_2} \right\rangle j_x = \langle E_x \rangle + \left\langle \frac{\sigma_1}{\sigma_2} \right\rangle E_y$$

Averaging Ohm's law in the same way over the remaining directions, we obtain the following expression for the effective conductivity tensor:

$$\begin{pmatrix} \left\langle \frac{1+(\Omega\tau)^2}{\sigma} \right\rangle^{-1} & \left\langle \frac{1+(\Omega\tau)^2}{\sigma} \right\rangle^{-1} \langle \Omega\tau \rangle & 0 \\ -\left\langle \frac{1+(\Omega\tau)^2}{\sigma} \right\rangle^{-1} \langle \Omega\tau \rangle & \langle \sigma \rangle - \langle \Omega\tau \rangle^2 \left\langle \frac{1+(\Omega\tau)^2}{\sigma} \right\rangle^{-1} & 0 \\ 0 & 0 & \langle \sigma \rangle \end{pmatrix}$$

If α is the angle between the resultant current and the x axis, we obtain for the conductivity along the current

$$\sigma_{eff} = \left(\frac{1}{\langle \sigma \rangle} \sin^2 \alpha + \left\langle \frac{1}{\sigma} \right\rangle \cos^2 \alpha + \Omega^2 \left(\left\langle \frac{\tau^2}{\sigma} \right\rangle - \frac{\langle \tau \rangle^2}{\langle \sigma \rangle} \right) \cos^2 \alpha \right)^{-1} \tag{9.24}$$

For small fluctuations, (9.24) yields

$$\sigma_{eff} = \sigma \left(1 + (\Omega\tau)^2 \left\langle \left(\frac{\Delta n}{n} \right)^2 \right\rangle \cos^2 \alpha \right)^{-1} \tag{9.25}$$

which is in agreement with (9.22). It follows from (9.22) that the maximum reduction in the conductivity arises from fluctuations whose wave vector is parallel to the lines of current flow.

The electrical conductivity of turbulent fully ionised plasma is calculated in [12] by a method analogous to that used above for the determination of the conductivity of weakly ionised, weakly turbulent plasma. Comparison is made with experiment, showing that the theory is in qualitative agreement with observation.

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Chapter 10

Dispersion of Electromagnetic Waves in Turbulent Plasma

The dispersion of electromagnetic waves in plasma is of interest both for the study of the properties of plasma created under laboratory conditions and for ionospheric studies. A number of papers have been devoted to the theory of dispersion of electromagnetic waves in laminar plasma (see, for example, [1]), and the results have been found to be in agreement with experiment. Theoretical investigation of dispersion of electromagnetic waves in weakly turbulent plasma may be carried out with the aid of kinetic equations for the wave distribution function (Chapter 7).

We shall derive equations describing the interaction of electromagnetic waves with weakly turbulent plasma, using the equations with self-consistent field and assuming that the frequency ω and wave vector $\mathbf{k}$ of the electromagnetic waves greatly exceed the frequency Ω and wave vector $\mathbf{q}$ of the turbulent pulsations respectively. The kinetic equation for the transverse waves distribution function n_k is thus

$$\frac{\partial n_k}{\partial t} + \mathbf{v}_k \frac{\partial n_k}{\partial \mathbf{x}} - \frac{\partial \omega_k}{\partial \mathbf{x}} \frac{\partial n_k}{\partial \mathbf{k}} = 0 \qquad (10.1)$$

where

$$\omega_k' = \omega_k + \mathbf{k}\cdot\mathbf{u}$$

($\omega_k = \sqrt{c^2k^2 + \omega_0^2}$, ω_0 being the Langmuir frequency, c the velocity of light and u the velocity of matter in the turbulent pulsations) is the frequency of the electromagnetic wave in the laboratory coordinate system and $\mathbf{k}\cdot\mathbf{u}$ is the Doppler frequency shift due to the presence of fluctuations in the velocity of matter.

Assuming that the main change in the frequency of transverse electromagnetic waves arises from a change in density and not from the velocity of matter in the turbulent pulsations (this is valid if $\omega^2_0/\omega \gg kv_\Phi$, where v_Φ is the phase velocity of the waves excited in the plasma), we obtain the following expression for the 'force' $\mathbf{f} = \frac{\partial\omega}{\partial\mathbf{x}}$ which acts on a quasi-particle (i.e. on a packet of transverse electromagnetic waves):

$$\mathbf{f} = -\frac{\omega_0^2}{\omega}\frac{\nabla n}{n} \tag{10.2}$$

Assuming further that the phases of the pulsations in turbulent plasma are random, and that these pulsations are weak, carrying out the operation of quasi-linear averaging in Equation (10.1), retaining the non-linear term $\overline{\nabla\omega' \frac{\partial n_\mathbf{k}}{\partial \mathbf{k}}}$ and substituting in it for $n_\mathbf{k}$ from the linearised Equation (10.1), we have

$$\dot{\bar{n}}_\mathbf{k} = \frac{\partial}{\partial k_\alpha} D_{\alpha\beta} \frac{\partial \bar{n}_\mathbf{k}}{\partial k_\beta} \tag{10.3}$$

where the diffusion coefficient for transverse waves in wave-vector space is given by

$$D_{\alpha\beta} = \sum_\mathbf{q}\left(\frac{\omega_0^2}{\omega}\right)^2 q_\alpha q_\beta \left|\frac{\delta n_q}{n}\right|^2 \delta(\Omega_q - \mathbf{k}\cdot\mathbf{v}_\mathbf{k}) \tag{10.4}$$

where δn_q is the fluctuation in the electron density in the turbulent plasma. We shall evaluate the diffusion coefficient D for isotropic turbulence and long-wavelength ion-sound oscillations:

$$D^s = SpD^s_{\alpha\beta} \approx \omega_0 \left(\frac{\omega_0}{c}\right)^2 \frac{\omega_0}{\omega} \frac{\langle q \rangle}{k} \left(\frac{\delta n^s}{n}\right)^2 \text{ when } \frac{c_s}{c} \frac{\omega}{ck} < 1$$
$$D^s = 0 \text{ when } \frac{c_s}{c} \frac{\omega}{ck} > 1 \tag{10.5}$$

Here $\langle q \rangle$ is the mean square of the wave number in the pulsations and δn^s is the mean square fluctuation of the electron density in the ion-sound waves.

It is important to note that for electromagnetic waves of frequency very close to the Langmuir frequency ($\omega/\omega_0 - 1 < (c_s/c)^2$), the diffusion coefficient D^s vanishes. The physical reason for this is that for these frequencies the group velocity of the tranverse waves, $\mathbf{v}_k = c c\mathbf{k}/\omega$, is smaller than the velocity of long-wavelength ion-sound waves c_s. There is little point, however, in considering the dispersion of electromagnetic waves of frequency very near to the Langmuir frequency ω_0 in terms of Equation (10.3), since in this case, even for weak density fluctuations, the character of the propagation of transverse perturbations undergoes a considerable change as the density passes through the critical value $n_{cr} = m\omega^2/4\pi e^2$: wave propagation is replaced by exponential damping.

Similarly, from (10.4) we have for isotropic turbulence arising from Langmuir oscillations [2]

$$D^l = S_p D^l_{\alpha\beta} \approx \omega_0 \left(\frac{\omega_0}{c}\right)^2 \frac{\omega_0}{\omega} \frac{\langle q \rangle}{k} \left(\frac{\delta n^l}{n}\right)^2 \tag{10.6}$$

where $\langle q \rangle$ is the mean square of the wave number in the density fluctuations and δn^l is the mean square fluctuation of electron density in the Langmuir oscillations. The diffusion coefficient D^l vanishes if oscillations with wavelength greater than $\frac{c}{\omega_0} \frac{c}{U}$ are absent from the Langmuir pulsation, spectrum (U is the phase velocity of transverse waves in the plasma).

Equation (10.3), or a more general equation of the type of (7.3) in the case of arbitrary relationship between q and k and Ω and ω, describes non-coherent dispersion of high-frequency electromagnetic waves. As an example, we shall evaluate with the aid of this equation the probability w of back-scattering of an electromagnetic wave crossing a layer of turbulent plasma of thickness l. Assuming that $w \ll 1$, we find from the diffusion equation (10.3)

$$\ln w \approx -\frac{\pi}{\langle\theta^2\rangle\, l} \tag{10.7}$$

where

$$\langle\theta^2\rangle \approx \frac{D}{k^2 v_k} \tag{10.8}$$

is the mean square displacement per unit path length of an electromagnetic beam from its initial direction, see also [3].

The frequency distribution of energy in transverse waves changes as they travel through turbulent plasma. This change may be used in plasma diagnostics.

Several aspects of the dispersion of electromagnetic waves in strongly turbulent plasma are considered in [3].

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Appendix A

Soft and Hard Modes of the Origin of Turbulence

The transition from laminar to turbulent state in a number of slightly supercritical systems is known to give rise to motion with well-defined frequency and wave number. Examples of this are fluids flowing between rotating cylinders, striations in gas discharges, screw instabilities in gas discharges and in semiconductors and convection between parallel surfaces. Motion which is periodic in time will also arise in a fluid flowing past solid objects. The frequency and wave number, and also the 'form' of the oscillations may be determined from the linear theory; to find the amplitude it is necessary to take non-linear effects into account.

The equation for the square of the modulus of the amplitude η of an unstable perturbation under conditions of small supercriticality is of the form

$$d\eta/dt = 2\eta(\gamma + a\eta + b\eta^2 + \ldots) = 2\eta\gamma_H$$

(the phase of the steady-state solution is arbitrary). Here a, b are functions of the parameters of the system (temperature, geometrical dimensions, electric and magnetic

fields), γ is the linear growth rate and the second and third terms in γ_H are related to non-linear effects.

The critical parameters λ_0 are defined by $\gamma(\lambda_0) = 0$; the equilibrium state $\eta = 0$ is unstable when $\gamma(\lambda) > 0$.

Let the system be such that $a(\lambda_0)$ is not zero for any value of λ_0. For $a(\lambda_0) < 0$, as the supercriticality $\lambda - \lambda_0$ increases, the amplitude of steady-state motion increases continuously from zero ('soft' mode); in this case we obtain from the equation $\gamma_H = 0$, $\eta = -\left(\frac{1}{a}\frac{\partial\gamma}{\partial\lambda}\right)_0 (\lambda - \lambda_0)$. For $a(\lambda_0) > 0$, as λ passes through the critical value of λ_0, the amplitude jumps from zero to some finite value ('hard' mode).

Consider a system for which $\gamma = a = 0$ holds for some values of the parameters λ_0. Let us select any two parameters λ and μ, and fix the remaining parameters so that the curves $a(\lambda, \mu) = 0$ and $\gamma(\lambda, \mu) = 0$ intersect. We measure λ, μ from the point of intersection; for a change in λ, the oscillations are 'soft' if $\mu < 0$, and 'hard' if $\mu > 0$.

Let $\Lambda = \lambda - \lambda_0$; for small values of $|\Lambda|$ and $|\mu|$, we can consider that $\gamma = \gamma'\Lambda$, $a = a'(\Lambda - \Lambda_0)$, $\Lambda_0 = c\mu$, where c and the derivatives γ', a' are taken at $\mu = \lambda = 0$. To be specific, let us take $\gamma', a' > 0$, $c < 0$. If $b \neq 0$ for $\lambda = \mu = 0$, it follows from $\gamma_H = 0$ that

$$\eta = A(\Lambda - \Lambda_0) \pm (A^2(\Lambda - \Lambda_0)^2 + B\Lambda)^{1/2}$$

$$A = -a'/2b, \quad B = -\gamma'/b$$

Consider the case when $b < 0$. A and B are then positive. Let Λ vary along the line $\mu = \text{const} > 0$ (here $\Delta_0 = c\mu < 0$). Then, for $\Lambda = +0$ the amplitude of the discontinuity varies from zero to $\eta_0 = -2A\Lambda_0 \backsim \mu$. If now Λ decreases, then for some value of $\Lambda = \delta < 0$ determined by the equation $A^2(\delta - \Lambda_0)^2 + B\delta = 0$, there is a jump in the amplitude from the value $\eta_\delta = A(\delta - \Lambda_0)$ to zero. Since $|\Lambda_0| \backsim \mu$ is a small quantity, $\delta \approx -(A\Lambda_0)^2/B = -(Ac)^2\mu^2/B \backsim \mu^2$; since $|\delta| \ll |\Lambda_0|$ for small μ, $\eta_\delta \approx -A\Lambda_0$ and

$$\eta_0/\eta_\delta = 2 \tag{A.1}$$

In the region $\delta < \Lambda < 0$ three steady-state solutions exist: $\eta = 0$ and $\eta_\pm = -A\Lambda_0(1 + \sqrt{1 - \Lambda/\delta})$; stable motions (observed experimentally) correspond to the values $\eta = 0$ and η_+. For $\delta < \Lambda < 0$ the system, originally in the state $\eta = 0$, may be made to go over to the state $\eta = \eta_+$ by imposing a perturbation of sufficiently large amplitude.

The slope of the curve $\eta = \eta(\Lambda)$ at the point $\Lambda = +0$ is equal to $d\eta/d\Lambda = D|\mu|$, $D > 0$. This expression is valid even for the soft mode.

As the amplitude of the stationary motion changes, the frequency ω and the average value of any observed quantity x will vary (for instance the average temperature, magnetic induction, electric current, particle flux, etc.). The corresponding functions for small-amplitude motions are of the form

$$x = x_p + x_1\eta + \ldots, \quad \omega = \omega_p + \omega_1\eta + \ldots \tag{A.2}$$

(If the steady-state motion is periodic in space, there is an analogous relation for the wave number: $k = k_p + k_1\eta + \ldots$) The quantities x, ω on the left-hand side of these equations are functions of Λ; x_p corresponds to the equilibrium state of the system $\eta = 0$ and ω_p is determined by the linear theory. Since $|\delta| \backsim \mu^2$, we have

$$\omega_p - \omega_\delta = (\omega_p + \omega_1\eta_0)_0 - (\omega_p + \omega_1\eta)\,\delta \approx \omega_1\,(\eta_0 - \eta_\delta) \backsim \mu$$

An analogous relation $x_0 - x_\delta \backsim \mu$ is obtained for the average value of an observable x in the presence of steady-state motion; if there is no such motion, $\eta_0 = \eta_\delta = 0$, then

$$x_0 - x_\delta = (x_p)_0 - (x_p)_\delta \backsim \delta \backsim \mu^2$$

For $\Lambda = 0$ and δ, the average value of any observable quantity changes discontinuously. Defining Δx as the amplitude of the discontinuity (the difference between the value of x in the presence of oscillations and in their absence for fixed Λ), we have for small μ by (A.1) and (A.2),

$$(\Delta x)_0/(\Delta x)_\delta = 2$$

Thus the amplitude of jumps in the average values and the squares of the varying parts, $x^2_\sim$, of observable quantities should satisfy the relation

$$(x^2_\sim)_0/(x_\sim)^2_\delta = (\Delta x)_0/(\Delta x)_\delta = 2 \tag{A.3}$$

The dependence of x on the parameter Λ may be of three types. When $\Lambda = 0$ one may observe (1) a sharp bend, (2) a discontinuity and (3) a radical singularity (case (3) corresponds to a change in Λ along the straight line $\mu = 0$).

Appendix B

Quasi-Linear Equations for a Quantum Plasma

As in the case of classical plasma, we will proceed from the equations with self-consistent field φ; then for the density matrix in the Wigner representation

$$f_{xp} = \sum_{\xi} \rho\left(x - \frac{\xi}{2},\ x + \frac{\xi}{2}\right)$$

where $\rho(y, z)$ satisfies the equation

$$\begin{aligned} i\,\frac{\partial \rho(y, z)}{\partial t} &= \left(-\frac{\Delta_y}{2} + \frac{\Delta_z}{2} + e\varphi(y) - e\varphi(z)\right)\rho(y, z) \\ &= \left(\nabla_x \nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right)\rho\left(x + \frac{\xi}{2},\ x - \frac{\xi}{2}\right) \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial f_{xp}}{\partial t} &= \frac{1}{i}\sum e^{-i\xi p}\left(\nabla_x \nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right) \\ &\quad \times \sum_q e^{i\xi q} f_{xq} = -p\,\frac{\partial f_{xp}}{\partial x} \\ &\quad + \frac{1}{i}\sum_q \left(e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right)\right) e^{i\xi(q-p)} f_{xq} \end{aligned} \tag{B.1}$$

We assume $\hbar = m = 1$ and for simplicity consider the isotropic spectrum $\varepsilon_p = p^2/2m$. Also for simplicity, we consider the case of longitudinal oscillations in an electron plasma with a positive 'background' of space charge. Equation (B.1) and Poisson's equation

$$\Delta_x \varphi = 4\pi ne \left(\sum f_{xp} - 1 \right) \tag{B.2}$$

are replaced in quasi-linear theory by a system of equations for the average value of the quantum distribution function $f^0 = \langle f_{xp} \rangle$, and for the deviation (oscillating with time) of the distribution function f_{xp} from its average value (the deviation is assumed small).

Choosing the oscillatory terms in (B.1) and (B.2) and transforming to the space Fourier components

$$\varphi(x) = \sum_k \varphi_k e^{ikx}, \quad f_{xp} - \langle f_{xp} \rangle = \sum_k f^1_{kp} e^{ikx} \tag{B.3}$$

we obtain for the spatially uniform case ($\nabla_x f^0 = 0$)

$$\dot{f}^1_{kp} + ikp f^1_{kp} = e\varphi_k \frac{f^0_{p+k/2} - f^0_{p-k/2}}{i}; \quad \varphi_k = \frac{4\pi ne}{k^2} \sum_p f^1_{kp} \tag{B.4}$$

On the other hand, averaging (B.1) over x, we have

$$\frac{\partial f^0_p}{\partial t} = i \sum_k e\varphi^+_k \left(f^1_{k,\, p+k/2} - f^1_{k,\, p-k/2} \right) \tag{B.5}$$

Further, integrating the ordinary differential equation (B.4) we obtain for the oscillatory part of the density matrix

$$f^1_{kp}(t) = f^1_{kp}(0)\, e^{-ikpt} + \int_0^t e\varphi_k(t') \frac{f^0_{p+k/2}(t') - f^0_{p-k/2}(t')}{i} e^{-ikp(t-t')} dt' \tag{B.6}$$

Differentiating (B.4) twice with respect to time, multiplying by $\dot{\varphi}^+_k$, combining this with its complex conjugate and substituting for f^1_{kp} from (B.6), we have

$$d(|\dot{\varphi}_k|^2 + \omega_k^2|\varphi_k^2|)/dt = \frac{4\pi n e^2}{k^2}\dot{\varphi}^+(t)\sum(kp)^2\left\{f_{kp}^1(0)\,e^{-ikpt}\right.$$
$$\left. + \int_0^t e\varphi_k(t')\frac{f_{p+k/2}^0 - f_{p-k/2}^0}{i}e^{-ikp(t-t')}\,dt'\right\} + \text{c.c.} \qquad \text{(B.7)}$$
$$\omega_k^2 = 4\pi n e^2\sum_p\frac{kp}{k^2}(f_{p+k/2}^0 - f_{p-k/2}^0) \simeq 4\pi n e^2$$

Substituting $\varphi_k(t) = |\varphi_k(t)|e^{-i\omega_k t}$ in (B.7), considering the slowly varying functions $|\varphi_k(t)|$ and $f^0(t)$ to be constant, taking them out from under the sign of integration over t' and considering that

$$\int_0^t e^{i\alpha(t-t')} + \text{c.c.} \underset{t\to\infty}{\rightarrow} 2\pi\delta(\alpha)$$

we obtain

$$\frac{d|\varphi_k^2|}{dt} = 4\pi n e^2|\varphi_k^2|\,\omega_k^{-1}\sum\frac{(kp)^2}{k^2}\left(f^0_{p+\frac{k}{2}} - f^0_{p-\frac{k}{2}}\right)\pi\delta(\omega_k - kp) \qquad \text{(B.8)}$$

Similarly, we obtain from (B.5), after substituting for f'_{kp} from (B.6),

$$\frac{df_p^0}{dt} = \pi e^2\sum_k|\varphi_k^2|\left\{(f_{p+k}^0 - f_p^0)\,\delta\left(\omega_k - k\left(p + \frac{k}{2}\right)\right)\right.$$
$$\left. - (f_p^0 - f_{p-k}^0)\,\delta\left(\omega_k - k\left(p - \frac{k}{2}\right)\right)\right\} \qquad \text{(B.9)}$$

Equations (B.8) and (B.9) form a complete set of quasi-linear equations for a quantum plasma.

Appendix C

Derivation of the Kinetic Equation Describing Three-plasmon Processes from the Hydrodynamic Equations

The system of equations of plasma hydrodynamics for quantities which do not deviate too greatly from their equilibrium values may be written in the form

$$i\frac{\partial\varphi}{\partial t} + H_0\varphi + H_1\{\varphi, \varphi\} = 0 \tag{C.1}$$

where φ is the state vector, represented by a column whose components are the velocity $\mathbf{v}$, the perturbations of the electric and magnetic fields $\mathbf{E}, \mathbf{H}$ and the perturbation of the density n; H_0 is a linear operator with real eigenvalues and H_1 is a bilinear vector operator.

The operator H_0 may be represented in the form of a matrix whose elements are differential operators; H_1 can be written in the form of a column vector formed by its components.

For small-amplitude waves, neglecting the non-linear term $H_1\{\varphi, \varphi\}$, we obtain the equation

$$i\frac{\partial\varphi}{\partial t} + H_0\varphi = 0 \tag{C.2}$$

This equation has eigenfunctions of the form $\varphi_k(\mathbf{x}t) = \varphi_k e^{-i(\omega_k t - kx)}$, where the frequency ω_k is real, because the operator H_0 is Hermitian. The vector φ may be expanded in terms of the eigenvectors of the operator H_0

$$\varphi = \sum_k (C_k^{(0)} \varphi_k e^{-i(\omega_k t -)} + C_k^{(0)} \varphi_k e^{i(\omega_k t -)}) \tag{C.3}$$

$$C_k^{(0)} = C_k^+, \quad \varphi_k = \varphi_k^+,$$

where $C_k^{(0)}$, $C_k^{(0)}$ are the complex amplitudes of the harmonics with wave vector k and frequencies $\omega_k, -\omega_k$, respectively, in the absence of interaction between separate waves.

The following postulate plays a basic role in the derivation of the kinetic equation: the phases of the amplitudes of the various waves $\alpha_k = \arg C_k$ are distributed completely randomly. This means that there is no correlation between the phases C_k in a time small compared with the time for a change in the quantity $|C_k^2|$ (i.e. the energy of the separate waves) because of the non-linear interaction between harmonics. This means that we may average over phases in deriving the kinetic equation

$$\langle C_k^{(0)} C_{k'}^{(0)} \rangle = |C_k^{(0)}|^2 \delta_{kk'} \tag{C.4}$$

We shall now divide the plasma into a slowly varying background and a rapidly oscillating part (the waves propagating in the plasma). The energy of these waves in the random-phase approximation is

$$\mathscr{E} = \sum_k \varepsilon_k \tag{C.5}$$

Normalising the state vector φ_k in agreement with the condition

$$|C_k|^2 = \frac{\varepsilon_k}{\omega_k} \tag{C.6}$$

we can interpret the square of the modulus of the wave amplitude $n_k = |C_k|^2$ as the number of quasi-particles with energy ω_ε. It is clear that the total energy and momentum of the background and of the quasi-particles together are conserved. Moreover, when the background changes slowly, the adiabatic invariant $n_k = \varepsilon_k / \omega_k$ (the energy of an oscillator divided by the frequency) is conserved for each quasi-particle:

$$\frac{D}{Dt}\frac{\varepsilon_k}{\omega_k} \equiv \frac{Dn_k}{Dt} = 0 \tag{C.7}$$

A change in the number of quasi-particles n_k in a given state due to collisions between them arises from the non-linear term $H_1(\varphi, \varphi)$ in (C.1). If this non-linear term is small, we may use perturbation theory. In the expansion (C.3), we must add to the state vector an orthogonal component φ'_k arising from the non-linear interaction

$$\varphi = \sum_k C_k(t)(\varphi_k + \varphi'_k)\, e^{-i(\omega_k t - kx)}\ \text{c.c.} \tag{C.8}$$

Substituting (C.8) into (C.1), we obtain to within second-order terms

$$-[\omega_k + H_0(k)]\, C_k\varphi'_k = i\frac{\partial C_k}{\partial t}\varphi_k + \sum_{k'+k''=k} C_{k'}C_{k''}H_1(k)\{\varphi_{k'}\varphi_{k''}\}\, e^{-i(\omega_{k'}+\omega_{k''}-\omega_k)t} \tag{C.9}$$

The condition that (C.9) is soluble is that the right-hand side must be orthogonal to the solution of the conjugate equation

$$\widetilde{\Psi}_k(\omega_k + H_0(k)) = 0 \tag{C.10}$$

where $\widetilde{\Psi}_k$ is a row vector.

Multiplying (C.9) scalarly on the left by $\widetilde{\Psi}$, we obtain

$$\left(\widetilde{\Psi}_k, \left[i\frac{\partial C_k}{\partial t}\varphi_k + \sum_{k'+k''=k} C_{k'}C_{k''}H_1\{\varphi_{k'}\varphi_{k''}\}\, e^{-i(\omega_{k'}+\omega_{k''}-\omega_k)t}\right]\right) = 0$$

Rewriting this equality in the form

$$\partial C_k/\partial t = -i\sum_{k'+k''=k} V_{kk'k''}C_{k'}C_{k''}\, e^{-i(\omega_{k'}+\omega_{k''}-\omega_k)t} \tag{C.11}$$

where

$$V_{kk'k''} = \frac{(\widetilde{\Psi}_k, H_1\{\varphi_{k'}, \varphi_{k''}\})}{(\widetilde{\Psi}_k, \varphi_k)}$$

we obtain the dynamic equation (C.11) in the k-representation.

To determine $C_k(t)$ at time t we use perturbation theory. From (C.11) we obtain

$$
\begin{aligned}
C_k(t) &= C_k^{(0)} + C_k^{(1)} + C_k^{(2)} + \dots \\
C_k^{(1)} &= -i \sum_{k'k''} C_{k'}^{(1)} C_{k''}^{(0)} \int_0^t V_{kk'k''}(t')\, dt' \\
C_k^{(2)} &= - \sum_{k'k''q'q''} C_{k'}^{(0)} C_{q'}^{(0)} C_{q''}^{(0)} \int_0^t dt' \int_0^{t'} dt'' V_{kk'k''}(t') V_{k''q'q''}(t'') \\
&\quad - \sum_{k'k''q'q''} C_{k''}^{(0)} C_{q'}^{(0)} C_{q''}^{(0)} \int_0^t dt' \int_0^{t'} dt'' V_{kk'k''}(t') V_{k'q'q''}(t'')
\end{aligned}
\qquad (C.12)
$$

where

$$
V_{kk'k''}(t) = V_{kk'k''} e^{-i(\omega_{k'}+\omega_{k''}-\omega_k)t}
$$
$$
V_{kk'k''} \neq 0 \text{ when } k' + k'' = k
$$

$C_k^{(0)}$ is independent of time and is a solution in the absence of interaction between harmonics.

The change in the number of quasi-particles $|C_k(t)|^2 - |C_k^{(0)}|^2$ averaged over the phases of $C_k^{(0)}$ with the aid of (C.4) is, to within second-order terms

$$
|C_k(t)|^2 - |C_k^{(0)}|^2 = \langle |C_k^{(1)}|^2 + (C_k^{(0)} C_k^{(1)} + C_k^{(0)} C_k^{(1)}) \rangle \qquad (C.13)
$$

Substituting (C.12), and remembering that

$$
\left| \int_0^t V_{kk'k''}(t)\, dt \right|^2 \to t \cdot 2\pi |V_{kk'k''}|^2 \delta(\omega_{k'} + \omega_{k''} - \omega_k) \quad \text{as} \quad t \to \infty \qquad (C.14)
$$

we obtain the change in the number of quasi-particles per unit time due to collisions:

$$
\begin{aligned}
\left(\frac{\partial n_k}{\partial t}\right)_s &= 4\pi \sum_{k'k''} |V_{kk'k''}|^2 \{ n_{k'} n_{k''} - n_k n_{k'} - n_k n_{k''}) \delta(\omega_{k'} + \omega_{k''} - \omega_k) \\
&\quad \times \delta(k' + k'' - k) + 2(n_{k'} n_{k''} + n_k n_{k''} - n_k n_{k'}) \\
&\quad \times \delta(\omega_{k''} - \omega_{k'} - \omega_k) \delta(k'' - k' - k)
\end{aligned}
\qquad (C.15)
$$

In Equation (C.15), summation extends over only those k', k'' for which $\omega_k, \omega_{k'}, \omega_{k''} > 0$. The first term in (C.15) describes the creation of a quasi-particle with energy ω_k as a result of the combination of quasi-particles with energies $\omega_{k'}$, $\omega_{k''}$ and the inverse process of decay. The second

term in (C.15) describes the decay of a quasi-particle with energy $\omega_{k''}$ into two quasi-particles with energies ω_k, $\omega_{k'}$ and the inverse process of combination.

The collision integral (C.15) vanishes for

$$n_k = \frac{T}{\omega_k}$$

where T is the effective 'temperature' of the quasi-particle gas, i.e. in the case when the quasi-particles are described by the Rayleigh-Jeans distribution.

Growth of Electrostatic Plasma Instability in a Strong Magnetic Field

Consider the growth of an electrostatic instability in fully ionised, tenuous plasma, assuming that the plasma is in a strong magnetic field H. The electric fields of the oscillations are potential and there is no perturbation of the magnetic field.

The quasi-linear equations may be obtained under these conditions by expanding the exact quasi-linear equations in $1/H$; they have the form

$$\partial f^{\alpha}/\partial t = \frac{\partial}{\partial v} D_{\parallel}^{\alpha} \frac{\partial f^{\alpha}}{\partial v} \tag{D.1}$$

$$D_{\parallel}^{\alpha} = \int d\mathbf{k} B^{\alpha}(\mathbf{k})\, \varepsilon(\mathbf{k})\, \delta(\omega_k - k_{\parallel} v) \tag{D.2}$$

$$\frac{\partial \varepsilon(\mathbf{k})}{\partial t} = \sum_{\alpha} A^{\alpha}(\mathbf{k})\, \varepsilon(\mathbf{k}) \int \frac{\partial f^{\alpha}}{\partial v}\, \delta(\omega_k - k_{\parallel} v) \tag{D.3}$$

where $f^{\alpha}(v) = \int F^{\alpha}(\mathbf{v})\, d\mathbf{v}_{\perp}$; $F^{\alpha}(\mathbf{v})$ is the velocity distribution function for particles of the type α, v is the projection of the velocity in the direction of the magnetic field, $v_{\perp}$ is the

perpendicular velocity component, $\varepsilon(\mathbf{k}) = \mathbf{E}_{\mathbf{k}}^2/8\pi$ is the spectral density of the electrostatic energy of the oscillations and ω_k is the frequency of the wave $\mathbf{k}$. The form of the functions A and B depends on the type of wave considered.

Equation (D.1) describes the diffusion of particles in velocity space under the influence of the plasma oscillations. The coefficient $D_{\parallel}$ is proportional to the energy of the plasma waves whose phase velocity $\omega_k/k_{\parallel}$ is equal to the velocity of the 'resonance particles' (Equation (D.2)). The change in the spectrum of oscillations is determined by the interaction of a given wave with the resonance particles (Equation (D.3)).

If the right-hand side of (D.3) is initially positive (i.e. there is a 'beam' in the plasma), the waves will grow. We shall consider oscillations for which 'skew' waves ($k_{\perp} \neq 0$) grow more slowly than purely longitudinal ones $(\partial^2 A_k/dk_{\perp}^2)_{k_{\perp}=0} = -A_1 < 0)$. The spectrum of oscillations developing in the plasma is then anisotropic, and the anisotropy increases with time. If the initial noise is sufficiently small, the spectrum becomes one-dimensional before any noticeable diffusion begins in phase space. Integrating (D.3) over time, we obtain

$$\varepsilon_k(t) = \varepsilon_k^0 \exp \int_0^t (A_0 - A_1 k_{\perp}^2 + \ldots) \frac{\partial f}{\partial v} dt \qquad \text{(D.4)}$$

where $A_0 = A$ $(k_{\perp} = 0)$. The integral $\int_0^t \frac{\partial f}{\partial v} dt$ increases monotonically with time. Thus for sufficiently large t, the higher terms of the expansion (D.4) become negligible and the distribution of energy over transverse wave numbers becomes Gaussian

$$\varepsilon_k(t) = C \exp - \frac{k_{\perp}^2}{\varkappa^2}, \quad C = \varepsilon_k^0 \exp \int_0^t A_0 \frac{\partial f}{\partial v} dt \qquad \text{(D.5)}$$

with the half-width

$$\varkappa(t) = \left(\int_0^t A_1 \frac{\partial f}{\partial v} dt \right)^{-1/2} \qquad \text{(D.6)}$$

Using (D.2) and (D.4) we obtain the magnitude of the coefficient of diffusion for particles in velocity space resulting

from interaction with the waves

$$D_{\parallel} = \pi \frac{B_0}{|v_\Gamma - v|} C\varkappa^2;\ B_0 = B(k_\perp = 0);\ v_\Gamma = \left(\frac{d\omega_k}{dk_\parallel}\right)_{k_\perp = 0} \tag{D.7}$$

The change in the diffusion coefficient with time is determined by the expression

$$\frac{\partial D_\parallel}{\partial t} = D_\parallel A_0 \frac{\partial f}{\partial v} \left\{ 1 - \frac{1}{\int A_0 \frac{\partial f}{\partial v} dt} + \dots \right\} \tag{D.8}$$

i.e. with time the integral $\left| \int A_0 \frac{\partial f}{\partial v} dt \right| \sim \ln \frac{\varepsilon}{\varepsilon_0}$ increases and the law for the change of the diffusion coefficient, (D.8), approximates to the law of change for a one-dimensional spectrum. With the aid of (D.6) and (D.7) we may relate the half-width of the spectrum $\varkappa(t)$ to the diffusion coefficient $D_\parallel$ or to the intensity of the plasma noise:

$$\varkappa(t) = \left(\frac{A_1}{A_0} \ln \frac{D_\parallel}{D^0_\parallel}\right)^{-1/2} = \left(\frac{A_1}{A_0} \ln \frac{\varepsilon}{\varepsilon^0}\right)^{-1/2} \tag{D.9}$$

Thus the equations describing the growth of an instability asymtotically approach the equations for a one-dimensional spectrum

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial}{\partial v} D_\parallel \frac{\partial f}{\partial v} \\ \frac{\partial D_\parallel}{\partial t} &= A_0 D_\parallel \frac{\partial f}{\partial v} \end{aligned} \tag{D.10}$$

Solving (D.10), we can find the noise intensity, the coefficient $D_\parallel(t)$ and the width of the energy distribution over the transverse wave numbers of the oscillations (Equation (D.9)).

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